

Height of k -coloured and k -generalised Motzkin paths

by Charlotte Brennan and Arnold Knopfmacher

University of the Witwatersrand
Johannesburg

SOUTH AFRICA

What is a Motzkin path?

- A Motzkin path is a lattice path in \mathbb{Z}^2 .

What is a Motzkin path?

- A Motzkin path is a lattice path in \mathbb{Z}^2 .
- Starts at $(0, 0)$ and ends at $(n, 0)$, always above the x -axis.

What is a Motzkin path?

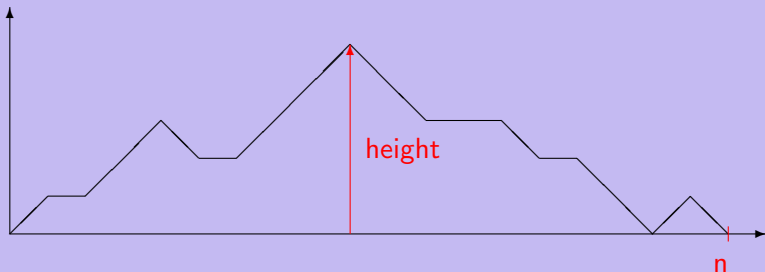
- A Motzkin path is a lattice path in \mathbb{Z}^2 .
- Starts at $(0, 0)$ and ends at $(n, 0)$, always above the x -axis.
- Allowed steps: up $(1, 1)$, horizontal $(1, 0)$, and down $(1, -1)$.

What is a Motzkin path?

- A Motzkin path is a lattice path in \mathbb{Z}^2 .
- Starts at $(0, 0)$ and ends at $(n, 0)$, always above the x -axis.
- Allowed steps: up $(1, 1)$, horizontal $(1, 0)$, and down $(1, -1)$.
- Height: maximum y -coordinate.

What is a Motzkin path?

- A Motzkin path is a lattice path in \mathbb{Z}^2 .
- Starts at $(0,0)$ and ends at $(n,0)$, always above the x -axis.
- Allowed steps: up $(1,1)$, horizontal $(1,0)$, and down $(1,-1)$.
- Height: maximum y -coordinate.



Two types of Motzkin paths

We consider two types of Motzkin paths:

- A k -coloured path:
The horizontal steps $(1, 0)$ are coloured with k colours.

Two types of Motzkin paths

We consider two types of Motzkin paths:

- A k -coloured path:
The horizontal steps $(1, 0)$ are coloured with k colours.
- A k -generalised path:
The horizontal steps are of length k , i.e., $(k, 0)$.
 $k = 1$ for *standard Motzkin* paths.
 $k = 0$ for *Dyck* paths.

Two types of Motzkin paths

We consider two types of Motzkin paths:

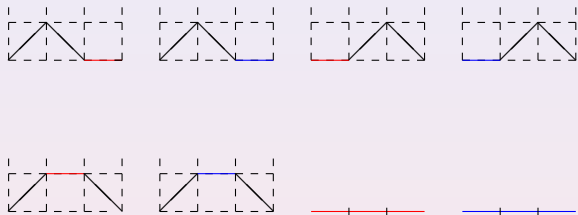
- A k -coloured path:
The horizontal steps $(1, 0)$ are coloured with k colours.
- A k -generalised path:
The horizontal steps are of length k , i.e., $(k, 0)$.
 $k = 1$ for *standard Motzkin* paths.
 $k = 0$ for *Dyck* paths.

Parameter: **TOTAL HEIGHT OF ALL THE PATHS** of length n .

The 14 Motzkin paths, $n = 3$, 2-coloured



The 14 Motzkin paths, $n = 3$, 2-coloured



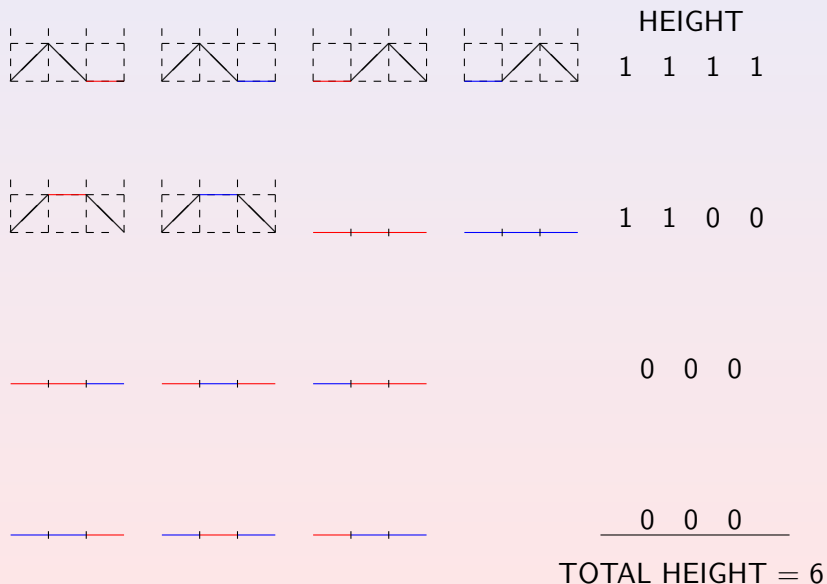
The 14 Motzkin paths, $n = 3$, 2-coloured



The 14 Motzkin paths, $n = 3$, 2-coloured



The 14 Motzkin paths, $n = 3$, 2-coloured

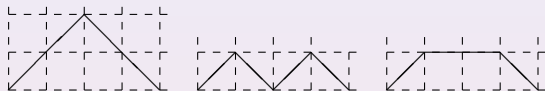


The 6 paths, $n = 4$, 2-generalised

The horizontal steps can only be multiples of $(2, 0)$,

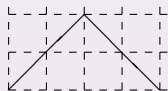
The 6 paths, $n = 4$, 2-generalised

The horizontal steps can only be multiples of $(2, 0)$,



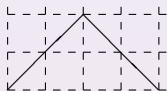
The 6 paths, $n = 4$, 2-generalised

The horizontal steps can only be multiples of $(2, 0)$,



The 6 paths, $n = 4$, 2-generalised

The horizontal steps can only be multiples of $(2, 0)$,



HEIGHT

2 1 1



1 1 0

TOTAL HEIGHT = 6

k -coloured paths: GENERATING FUNCTIONS

Let $\phi_j(y)$ be the generating function for paths with height $\leq m$, ending at (n, j) , where y counts the unit steps.

k -coloured paths: GENERATING FUNCTIONS

Let $\phi_j(y)$ be the generating function for paths with height $\leq m$, ending at (n, j) , where y counts the unit steps.

Recurrences

$$\phi_0(y) = 1 + yk\phi_0(y) + y\phi_1(y) \quad \text{for } j = 0$$

$$\phi_j(y) = y\phi_{j-1}(y) + ky\phi_j(y) + y\phi_{j+1}(y) \quad \text{for } 0 < j < m$$

$$\phi_m(y) = ky\phi_m(y) + y\phi_{m-1}(y) \quad \text{for } j = m.$$

k -coloured paths: GENERATING FUNCTIONS

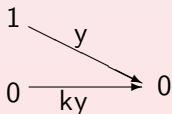
Let $\phi_j(y)$ be the generating function for paths with height $\leq m$, ending at (n, j) , where y counts the unit steps.

Recurrences

$$\phi_0(y) = 1 + yk\phi_0(y) + y\phi_1(y) \quad \text{for } j = 0$$

$$\phi_j(y) = y\phi_{j-1}(y) + ky\phi_j(y) + y\phi_{j+1}(y) \quad \text{for } 0 < j < m$$

$$\phi_m(y) = ky\phi_m(y) + y\phi_{m-1}(y) \quad \text{for } j = m.$$



k -coloured paths: GENERATING FUNCTIONS

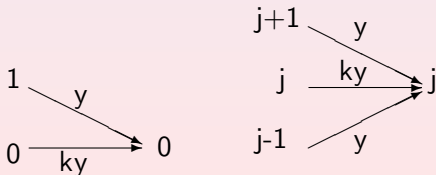
Let $\phi_j(y)$ be the generating function for paths with height $\leq m$, ending at (n, j) , where y counts the unit steps.

Recurrences

$$\phi_0(y) = 1 + yk\phi_0(y) + y\phi_1(y) \quad \text{for } j = 0$$

$$\phi_j(y) = y\phi_{j-1}(y) + ky\phi_j(y) + y\phi_{j+1}(y) \quad \text{for } 0 < j < m$$

$$\phi_m(y) = ky\phi_m(y) + y\phi_{m-1}(y) \quad \text{for } j = m.$$



k -coloured paths: GENERATING FUNCTIONS

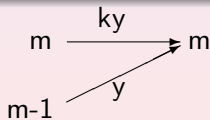
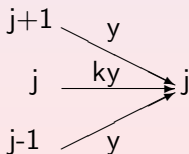
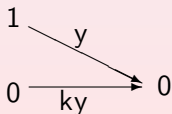
Let $\phi_j(y)$ be the generating function for paths with height $\leq m$, ending at (n, j) , where y counts the unit steps.

Recurrences

$$\phi_0(y) = 1 + yk\phi_0(y) + y\phi_1(y) \quad \text{for } j = 0$$

$$\phi_j(y) = y\phi_{j-1}(y) + ky\phi_j(y) + y\phi_{j+1}(y) \quad \text{for } 0 < j < m$$

$$\phi_m(y) = ky\phi_m(y) + y\phi_{m-1}(y) \quad \text{for } j = m.$$



Recurrences in a matrix

Let $\phi_j(y) := \phi_j$, for paths ending at (n, j) .

The $m + 1$ equations can be represented in a matrix:

$$\begin{pmatrix} 1 - ky & -y & 0 & \dots & 0 \\ -y & 1 - ky & -y & & \\ 0 & -y & 1 - ky & & \\ \vdots & & & \ddots & \vdots \\ & & & & 1 - ky & -y & 0 \\ & & & & -y & 1 - ky & -y \\ 0 & \dots & & 0 & -y & 1 - ky \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{m-2} \\ \phi_{m-1} \\ \phi_m \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since the Motzkin paths end on the x -axis, $j = 0$, we are interested in $\phi_0(y)$.

$$\phi_0(y) := A_{\leq m}(y)$$

We rename $\phi_0(y)$ with $A_{\leq m}(y)$, the generating function for all Motzkin paths of length n , ending at $(n, 0)$, with height at most m , and with horizontal steps coloured with k colours.

$$\phi_0(y) := A_{\leq m}(y)$$

We rename $\phi_0(y)$ with $A_{\leq m}(y)$, the generating function for all Motzkin paths of length n , ending at $(n, 0)$, with height at most m , and with horizontal steps coloured with k colours.

Then, $[y^n]A_{\leq m}(y)$ is the number of such paths of length n .

$$\phi_0(y) := A_{\leq m}(y)$$

We rename $\phi_0(y)$ with $A_{\leq m}(y)$, the generating function for all Motzkin paths of length n , ending at $(n, 0)$, with height at most m , and with horizontal steps coloured with k colours.

Then, $[y^n]A_{\leq m}(y)$ is the number of such paths of length n .

Using Cramer's rule, we have

$$\phi_0(y) = 2 \frac{\left(1 - ky + \sqrt{X}\right)^{m+1} - \left(1 - ky - \sqrt{X}\right)^{m+1}}{\left(1 - ky + \sqrt{X}\right)^{m+2} - \left(1 - ky - \sqrt{X}\right)^{m+2}}$$

where

$$X = 1 - 2ky - 4y^2 + k^2y^2.$$

After the substitution $y = \frac{v}{1+kv+v^2}$, we have a simpler expression:

$A_{\leq m}(y)$

$$A_{\leq m}(y) = \frac{(1 + kv + v^2)(1 - v^{2m+2})}{1 - v^{2m+4}}.$$

$A_{\leq m}(y)$

$$A_{\leq m}(y) = \frac{(1 + kv + v^2)(1 - v^{2m+2})}{1 - v^{2m+4}}.$$

Assuming $|v| < 1$, letting $m \rightarrow \infty$, we have the g.f. for ALL k -coloured Motzkin paths

 $A_{\infty}(y)$

$$A_{\infty}(y) = 1 + kv + v^2.$$

$A_{\leq m}(y)$

$$A_{\leq m}(y) = \frac{(1 + kv + v^2)(1 - v^{2m+2})}{1 - v^{2m+4}}.$$

Assuming $|v| < 1$, letting $m \rightarrow \infty$, we have the g.f. for ALL k -coloured Motzkin paths

 $A_{\infty}(y)$

$$A_{\infty}(y) = 1 + kv + v^2.$$

So

 $A_{> m}(y)$

$$A_{> m}(y) = A_{\infty}(y) - A_{\leq m}(y) = \frac{(1 + kv + v^2)v^{2m+2}(1 - v^2)}{1 - v^{2m+4}}.$$

Thus the generating function for the TOTAL HEIGHT is:

$E(y)$

$$\begin{aligned} & \sum_{m=0}^{\infty} A_{>m}(y) \\ &= \sum_{m=0}^{\infty} \frac{(1 + kv + v^2)v^{2m+2}(1 - v^2)}{1 - v^{2m+4}} \\ &= (1 + kv + v^2) \frac{1 - v^2}{v^2} \sum_{n \geq 1} (d(n) - 1)v^{2n}. \end{aligned}$$

Thus the generating function for the TOTAL HEIGHT is:

$E(y)$

$$\begin{aligned} & \sum_{m=0}^{\infty} A_{>m}(y) \\ &= \sum_{m=0}^{\infty} \frac{(1 + kv + v^2)v^{2m+2}(1 - v^2)}{1 - v^{2m+4}} \\ &= (1 + kv + v^2) \frac{1 - v^2}{v^2} \sum_{n \geq 1} (d(n) - 1)v^{2n}. \end{aligned}$$

Series expansion for $k = 2$ is

$$y^2 + 6y^3 + 27y^4 + 110y^5 + 430y^6 + 1652y^7 + 6306y^8 + \dots$$

Aim: To find an asymptotic expression for $[y^n]E(y)$, the total height for paths of length n as $n \rightarrow \infty$. We analyse the variable by means of an integral representation together with residue calculus and contour integration:

Aim: To find an asymptotic expression for $[y^n]E(y)$, the total height for paths of length n as $n \rightarrow \infty$. We analyse the variable by means of an integral representation together with residue calculus and contour integration:

Mellin transform

$$f^*(s) = \int_0^{\infty} t^{s-1} f(t) dt.$$

Aim: To find an asymptotic expression for $[y^n]E(y)$, the total height for paths of length n as $n \rightarrow \infty$. We analyse the variable by means of an integral representation together with residue calculus and contour integration:

Mellin transform

$$f^*(s) = \int_0^\infty t^{s-1} f(t) dt.$$

Inverse Mellin transform

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-s} f^*(s) ds,$$

where c lies in the vertical strip of convergence of $f^*(s)$.

With the use of the substitution $v = e^{-t}$

$$\begin{aligned}
 E(y) &= (1 + kv + v^2) \frac{1 - v^2}{v^2} \sum_{n \geq 1} (d(n) - 1) v^{2n} \\
 &= \underbrace{(1 + ke^{-t} + e^{-2t})(1 - e^{-2t})e^{2t}}_{A(t)} \underbrace{\sum_{n \geq 1} (d(n) - 1) e^{-2nt}}_{B(t)}
 \end{aligned}$$

where

$$B^*(s) = \sum_{n \geq 1} (d(n) - 1) (2n)^{-s} \Gamma(s) = \zeta(s) (\zeta(s) - 1) 2^{-s} \Gamma(s).$$

After applying the Inverse Mellin transform, and expanding around the dominant singularities, we have

$$\begin{aligned} E(y) \sim & -2 \ln 2 - k \ln 2 - 2 \ln t - k \ln t - 2 + 2\gamma - k + \gamma k \\ & + \frac{t}{2}(6 + 3k) \\ & + \frac{t^2}{36}(-74 + 48\gamma - 19k + 6\gamma k - 48 \ln 2 - 6k \ln 2 - 48 \ln t - 6k \ln t) \\ & + \frac{t^3}{4}(8 + k). \end{aligned}$$

After substituting t as a function of y , finding the coefficient of y^n and dividing by the total number of Motzkin paths of size n , we have

Theorem 1:

The average total height of Motzkin paths with horizontal steps with $k \geq 1$ colours is as $n \rightarrow \infty$

$$e_k(n) \sim \frac{\sqrt{n\pi}}{\sqrt{k+2}} - \frac{3}{2} + \frac{(k+44)\sqrt{\pi}}{48\sqrt{k+2}\sqrt{n}}.$$

Let $\phi_{j,k}(y)$ be the generating function that counts the paths with height $\leq m$ and ending at the point (n, j) . We want $\phi_{0,k}(y)$.

Recurrences

$$\phi_{0,k}(y) = 1 + y^k \phi_{0,k}(y) + y \phi_{1,k}(y) \quad \text{for } j = 0$$

$$\phi_{j,k}(y) = y \phi_{j-1,k}(y) + y^k \phi_{j,k}(y) + y \phi_{j+1,k}(y) \quad \text{for } 0 < j < m$$

$$\phi_{m,k}(y) = y^k \phi_{m,k}(y) + y \phi_{m-1,k}(y) \quad \text{for } j = m.$$

The $m + 1$ equations in a matrix form

Again, we put the $m + 1$ equations in a matrix form.

The square matrix is:

Matrix

$$\begin{pmatrix} 1 - y^k & -y & 0 & \dots & 0 \\ -y & 1 - y^k & -y & & \\ 0 & -y & 1 - y^k & & \\ \vdots & & & \ddots & \vdots \\ & & & & 1 - y^k & -y & 0 \\ & & & & -y & 1 - y^k & -y \\ 0 & \dots & 0 & -y & 1 - y^k \end{pmatrix}$$

The determinant $a_{m,k}(y)$ of that square matrix is

$$a_{m,k}(y) = \frac{\left(\frac{1-y^k+\sqrt{X}}{2}\right)^{m+2} - \left(\frac{1-y^k-\sqrt{X}}{2}\right)^{m+2}}{\sqrt{X}},$$

where

$$X = 1 - 2y^k - 4y^2 + y^{2k}.$$

The determinant $a_{m,k}(y)$ of that square matrix is

$$a_{m,k}(y) = \frac{\left(\frac{1-y^k+\sqrt{X}}{2}\right)^{m+2} - \left(\frac{1-y^k-\sqrt{X}}{2}\right)^{m+2}}{\sqrt{X}},$$

where

$$X = 1 - 2y^k - 4y^2 + y^{2k}.$$

With Cramer's rule and the substitution $y^k = \frac{v-y-yv^2}{v}$, we get

$$\phi_{0,k}(y) = A_{\leq m,k}(y) = \frac{v(1 - v^{2m+2})}{y(1 - v^{2m+4})},$$

which as $m \rightarrow \infty$ gives $A_{\infty,k}(y) = \frac{v}{y}$.

After performing $A_{\infty,k}(y) - A_{\leq m,k}(y)$ to get $A_{> m,k}(y)$ we have

$$E(y) = \sum_{m \geq 0} A_{> m,k}(y) = \frac{v(1-v^2)}{y} \sum_{m \geq 1} (d(n) - 1)v^{2n}.$$

All Motzkin k -generalised paths

Using the first return decomposition:

Symbolic equation

$$\cdot \quad \text{OR} \quad \text{---} \mathcal{M} \quad \text{OR} \quad \begin{array}{c} \mathcal{M} \\ \diagdown \quad \diagup \\ \mathcal{M} \end{array}$$
$$1 \quad \quad \quad y^k M \quad \quad \quad y^2 M^2.$$

This translates to

$$M(y) = 1 + y^k M(y) + y^2 M(y)^2,$$

or

$$M(y) = \frac{1 - y^k - \sqrt{(1 - y^k)^2 - 4y^2}}{2y^2} = \frac{v}{y}.$$

Asymptotics for the number of all the generalised paths

To get asymptotics for the total height, we use singularity analysis and expand around the dominant singularity ρ , the root of $(1 - y^k)^2 - 4y^2 = 0$. For k odd,

Theorem 2: Asymptotics for number of generalised paths

$$M_k(n) \sim \rho^{-n} \left[\frac{\sqrt{\rho(8\rho + 4k\rho^k)}}{4\sqrt{\pi}\rho^2 n^{3/2}} + \frac{3P}{64\sqrt{\pi}\rho^4 \sqrt{\rho(8\rho + 4k\rho^k)} n^{5/2}} \right]$$

where P is

$$14\rho^3(-8\rho - 4k\rho^k) + 4\rho^3(\rho(-8\rho - 4k\rho^k)^2 - \rho(8 + 4(1 - k)k\rho^{-1+k} + 2k^2\rho^{-2+2k})).$$

Asymptotics: Total height of k -generalised paths, k odd

Applying the Mellin transform, with the substitution $v = e^{-t}$:

Theorem 3: Asymptotics for Total height

$$T_k(n) := \rho^{-(n+1)} \left[\frac{1}{2n} - \frac{3\sqrt{2 + k\rho^{-1+k}}}{4\sqrt{\pi}n^{3/2}} - \frac{4\rho - k\rho^k}{12\rho n^2} \right].$$

Dividing $T_k(n)$ by $M_k(n)$:

Theorem 4: Mean height, k odd

$$e_k(n) \sim \frac{\sqrt{\pi n \rho}}{\sqrt{\rho(2\rho + k\rho^k)}} - \frac{3}{2} - \frac{\sqrt{\pi} Q}{24(2\rho + k\rho^k)^{3/2} \sqrt{n\rho}},$$

where

$$Q = -130\rho^2 + 288\rho^4 - 13k^2\rho^{2k} - 73k\rho^{1+k} + 18k^2\rho^{1+k} + 288k\rho^{3+k} + 72k^2\rho^{2+2k}.$$

MERCI

Pour les francophones et les organisateurs de cette conférence, je vous remercie et je vous souhaite une bonne semaine!