

Fixed point equations
for probability distributions
coming from combinatorics

Brigitte CHAUVIN

(Univ. Versailles)

with Quansheng LIU (Vannes)

Cécile MAILLER (Versailles)

and Nicolas POUYANNE (Versailles)

Gascom 2012

Outline

Discrete structures

Markov chains

recursion

auto similarity

divide and conquer

trees

continuous time
branching processes

branching property

martingales

splitting equations

$\downarrow n \rightarrow +\infty$

fixed point equations

\rightarrow Pólya urns

\rightarrow m -ary search trees

$\downarrow t \rightarrow +\infty$

\mathbb{R} -valued solutions

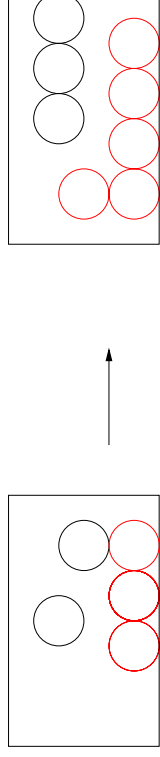
\mathbb{C} -valued solutions

unicity, support, density, moments,...

Part I

Pólya urn - Discrete time

Pólya-Eggenberger urns with 2 colours



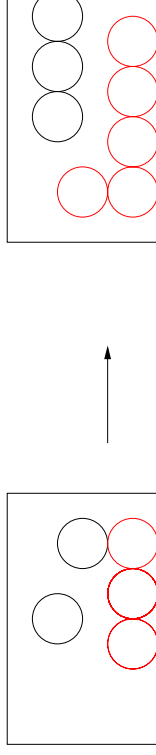
Replacement matrix $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, a, b, c, d integers.

Composition vector $U^{DT}(n) = \begin{pmatrix} \# \text{ red at time } n \\ \# \text{ black at time } n \end{pmatrix}$;

$$U^{DT}(0) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Question: Asymptotics of $U^{DT}(n)$?

Pólya urn - Discrete time



Replacement matrix $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, a, b, c, d integers.

$$U^{DT}(n) = \begin{pmatrix} \# \text{ red at time } n \\ \# \text{ black at time } n \end{pmatrix}; \quad U^{DT}(0) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Assumption

at time $n = \alpha + \beta + Sn$.

Balance: $a + b = c + d := S$

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} S & 0 \\ 0 & m \end{pmatrix} \quad m = \sigma S = a - c = d - b \quad \boxed{\sigma = \frac{m}{S}}$$

Original Pólya urn $R = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}$ $m = S$ $\sigma = \frac{m}{S} = 1$

Th (Pólya, '30, Guet, '97)

Starting from (α, β) ,

$$\frac{U_{(\alpha, \beta)}^{DT}(n)}{nS} \xrightarrow[n \rightarrow \infty]{} V \quad a.s.$$

where $V = (V_1, V_2) \stackrel{\mathcal{L}}{=} \text{Dirichlet}(\frac{\alpha}{S}, \frac{\beta}{S})$:

$$\begin{cases} V_1 + V_2 = 1 \\ V_1 \stackrel{\mathcal{L}}{=} \text{Beta}(\frac{\alpha}{S}, \frac{\beta}{S}) \quad V_2 \stackrel{\mathcal{L}}{=} \text{Beta}(\frac{\beta}{S}, \frac{\alpha}{S}) \end{cases}$$

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} S & 0 \\ 0 & m \end{pmatrix} \quad m = \sigma S = a - c = d - b$$

$\sigma \neq 1$
 $bc \neq 0$

$\sigma = \frac{m}{S}$

2 kinds of asymptotics:

- Gaussian: when $\sigma < \frac{1}{2}$ “small” urn

$$\frac{U^{DT}(n) - nv_1}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \Sigma^2)$$

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} S & 0 \\ 0 & m \end{pmatrix} \quad m = \sigma S = a - c = d - b \quad \sigma = \frac{m}{S}$$

2 kinds of asymptotics:

- Gaussian: when $\sigma \leq \frac{1}{2}$ “small” urn

$$\frac{U^{DT}(n) - nv_1}{\sqrt{n} \sqrt{\log n}} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \Sigma^2)$$

2 kinds of asymptotics:

- Gaussian: when $\sigma < \frac{1}{2}$ “small” urn

$$\frac{U^{DT}(n) - nv_1}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \Sigma^2)$$

- non Gaussian: when $\frac{1}{2} < \sigma < 1$ “large” urn

$$U^{DT}(n) = nv_1 + n^\sigma W^{DT} v_2 + o(n^\sigma)$$

where

v_1, v_2 are deterministic vectors

W^{DT} is a martingale’s limit

$o(\cdot)$ means a.s. and in any $L^p, p \geq 1$

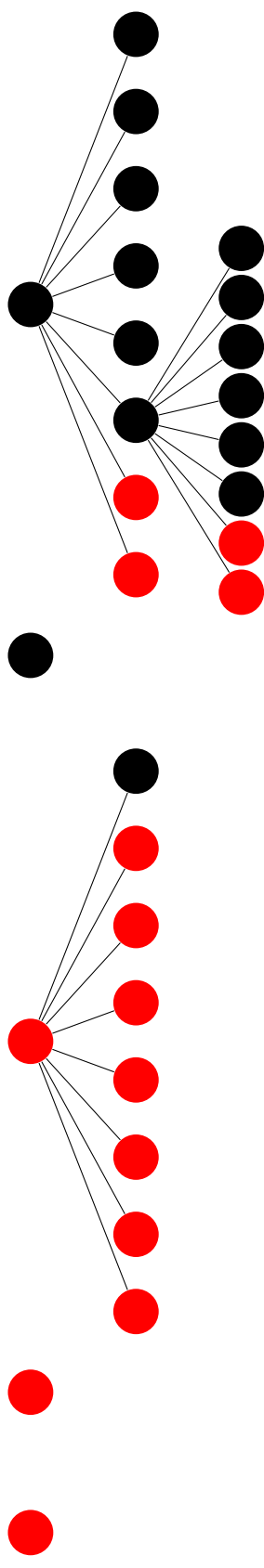
the moments of W^{DT} can be recursively calculated.

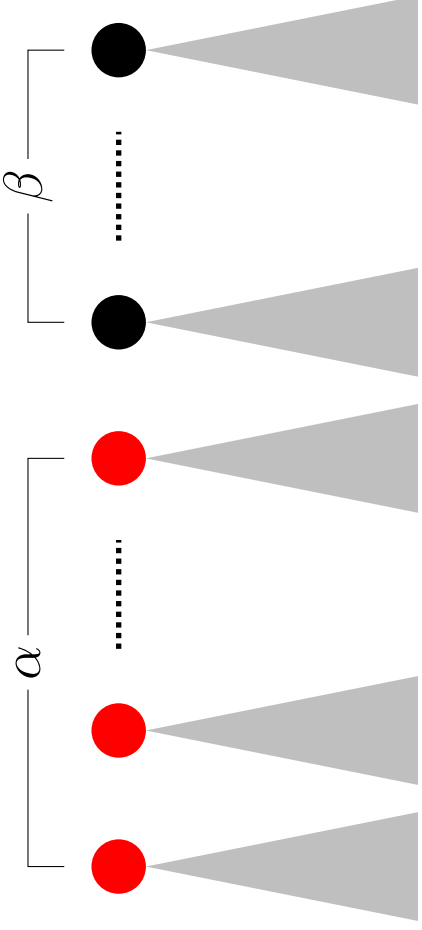
Goal: **distribution of W^{DT}**

Splitting \leftarrow Tree structure

$$U_{(\alpha,\beta)}^{DT}(n) = nv_1 + n^\sigma W_{(\alpha,\beta)}^{DT} v_2 + o(n^\sigma)$$

Ex: $R = \begin{pmatrix} 6 & 1 \\ 2 & 5 \end{pmatrix}, \quad \alpha = 3, \beta = 2$





$$U_{(\alpha,\beta)}(n) \stackrel{\mathcal{L}}{=} \sum_{k=1}^{\alpha} U_{(1,0)}^{(k)} \left(\frac{D_k(n) - 1}{S} \right) + \sum_{k=1+\alpha}^{\alpha+\beta} U_{(0,1)}^{(k)} \left(\frac{D_k(n) - 1}{S} \right)$$

$D_k(n) := \text{Card}(\text{subtree of the descendants from the } k\text{-th initial ball at time } n)$

$D_k(n) - 1$ is a multiple of S .

Splitting

$$U_{(\alpha,\beta)}(n) \stackrel{\mathcal{L}}{=} \sum_{k=1}^{\alpha} U_{(1,0)}^{(k)} \left(\frac{D_k(n) - 1}{S} \right) + \sum_{k=1+\alpha}^{\alpha+\beta} U_{(0,1)}^{(k)} \left(\frac{D_k(n) - 1}{S} \right)$$

$(D_1(n), \dots, D_{\alpha+\beta}(n))$ is an “original Pólya” urn with replacement matrix $SI_{\alpha+\beta}$.

$$(D_1(n), \dots, D_{\alpha+\beta}(n)) \xrightarrow[n \rightarrow \infty]{} \text{Dirichlet} \left(\frac{1}{S}, \dots, \frac{1}{S} \right) \quad a.s.$$

$$U_{(\alpha,\beta)}(n) \stackrel{\mathcal{L}}{=} \sum_{k=1}^{\alpha} U_{(1,0)}^{(k)} \left(\frac{D_k(n) - 1}{S} \right) + \sum_{k=1+\alpha}^{\alpha+\beta} U_{(0,1)}^{(k)} \left(\frac{D_k(n) - 1}{S} \right)$$

$n \rightarrow +\infty$

$$U_{(\alpha,\beta)}^{DT}(n) = nv_1 + n^\sigma W_{(\alpha,\beta)}^{DT} v_2 + o(n^\sigma)$$

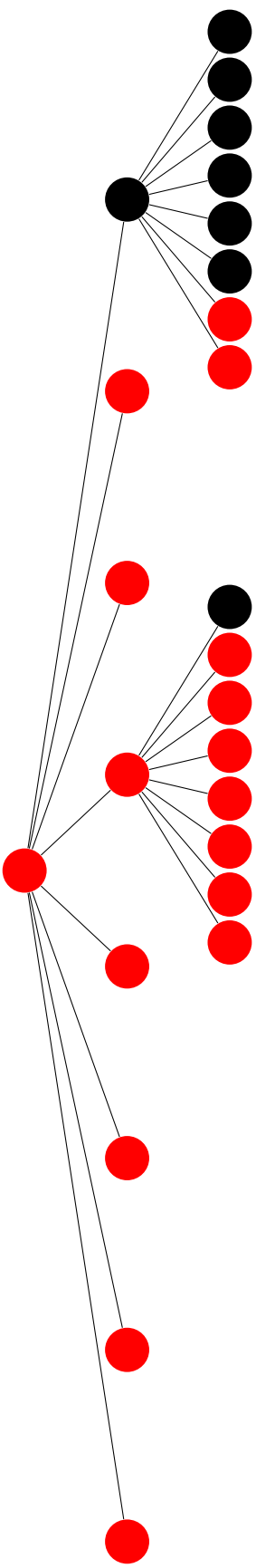
$$W_{(\alpha,\beta)} \stackrel{\mathcal{L}}{=} \sum_{k=1}^{\alpha} (V_k)^\sigma W_{(1,0)}^{(k)} + \sum_{k=\alpha+1}^{\alpha+\beta} (V_k)^\sigma W_{(0,1)}^{(k)}$$

where

$$(V_1, \dots, V_{\alpha+\beta}) \stackrel{\mathcal{L}}{=} \text{Dirichlet} \left(\frac{1}{S}, \dots, \frac{1}{S} \right) :$$

$$\left\{ \begin{array}{l} \sum_{k=1}^{\alpha+\beta} V_k = 1 \\ V_k \stackrel{\mathcal{L}}{=} \text{Beta} \left(\frac{1}{S}, \frac{\alpha + \beta - 1}{S} \right) \end{array} \right.$$

$$\text{Ex: } R = \begin{pmatrix} 6 & 1 \\ 2 & 5 \end{pmatrix}$$



$$U_{(1,0)}(n) \stackrel{\mathcal{L}}{=} \sum_{k=1}^{a+1} U_{(1,0)}^{(k)} \left(\frac{J_k(n) - 1}{S} \right) + \sum_{k=a+2}^{S+1} U_{(0,1)}^{(k)} \left(\frac{J_k(n) - 1}{S} \right)$$

$$n \rightarrow +\infty \quad W_{(1,0)} \stackrel{\mathcal{L}}{=} \sum_{k=1}^{a+1} (V_k)^\sigma W_{(1,0)}^{(k)} + \sum_{k=a+2}^{S+1} (V_k)^\sigma W_{(0,1)}^{(k)}$$

$$\text{notation } \begin{cases} X := W_{(1,0)} \\ Y := W_{(0,1)} \end{cases} \quad X \stackrel{\mathcal{L}}{=} \sum_{k=1}^{a+1} (V_k)^\sigma X^{(k)} + \sum_{k=a+2}^{S+1} (V_k)^\sigma Y^{(k)}$$

$$\left\{ \begin{array}{l} X \stackrel{\mathcal{L}}{=} \sum_{k=1}^{a+1} (V_k)^\sigma X^{(k)} + \sum_{k=a+2}^{S+1} (V_k)^\sigma Y^{(k)} \\ Y \stackrel{\mathcal{L}}{=} \sum_{k=1}^c (V_k)^\sigma X^{(k)} + \sum_{k=c+1}^{S+1} (V_k)^\sigma Y^{(k)} \end{array} \right.$$

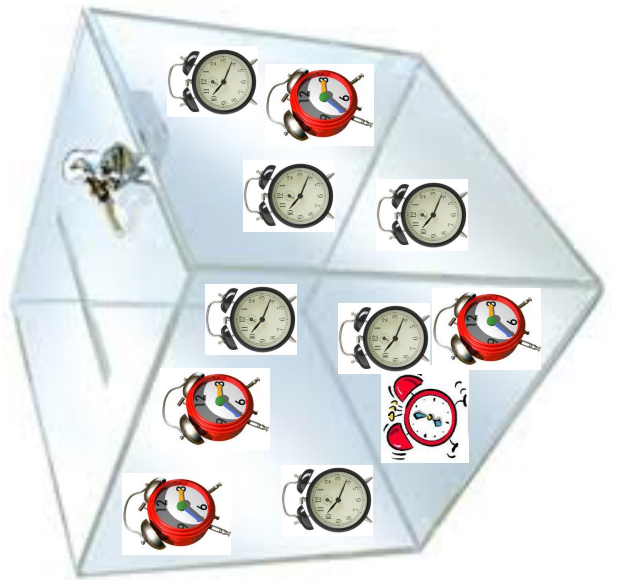
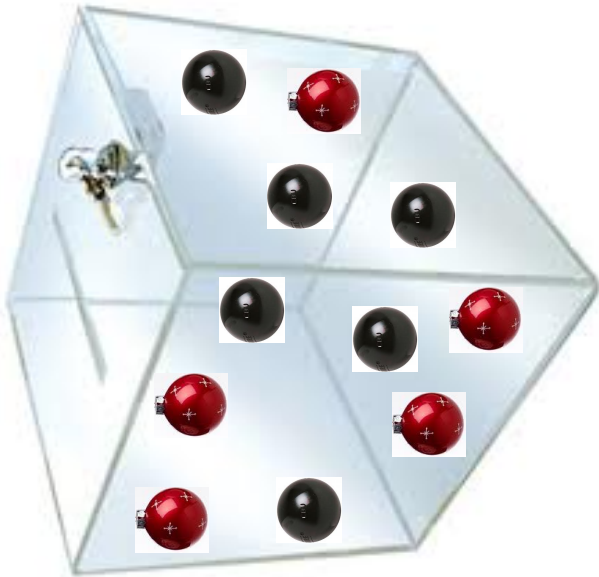
where

$$(V_1, \dots, V_{S+1}) \stackrel{\mathcal{L}}{=} \text{Dirichlet} \left(\frac{1}{S}, \dots, \frac{1}{S} \right) :$$

$$\left\{ \begin{array}{l} \sum_{k=1}^{S+1} V_k = 1 \\ V_k \stackrel{\mathcal{L}}{=} \text{Beta} \left(\frac{1}{S}, 1 \right) = (\text{Unif})^S \quad (V_k)^\sigma = (\text{Unif})^m \end{array} \right.$$

Part II

Pólya urn - Continuous time



Continuous-time Markov process $(U^{CT}(t))_{t \geq 0}$

$$U^{CT}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Exp(1) clock on each ball; independence of the clocks.

Branching:

- if a red ball rings, replace it by $a + 1$ red balls and b black balls
- if a black ball rings, replace it by c red balls and $d + 1$ black balls

$0 < \tau_1 < \dots < \tau_n < \dots$ are the jump times

$$(U^{CT}(\tau_n))_{n \geq 0} \stackrel{\mathcal{L}}{=} (U^{DT}(n))_{n \geq 0}$$

Benefit: **independence**

if $\sigma > 1/2$,

$$U_{(\alpha,\beta)}^{CT}(t) = e^{St} \xi v_1 (1 + o(1)) + e^{mt} W_{(\alpha,\beta)}^{CT} v_2 (1 + o(1))$$

where $\xi \stackrel{\mathcal{L}}{=} \text{Gamma}(\frac{\alpha+\beta}{S})$.

branching property

$$W_{(\alpha,\beta)}^{CT} \stackrel{\mathcal{L}}{=} \sum_{k=1}^{\alpha} W_{(1,0)}^{CT}(k) + \sum_{k=\alpha+1}^{\alpha+\beta} W_{(0,1)}^{CT}(k)$$

$$\text{notation} \begin{cases} X := W_{(1,0)}^{CT} \\ Y := W_{(0,1)}^{CT} \end{cases}$$

branching property again (...)

$$\begin{cases} X \stackrel{\mathcal{L}}{=} U^m \left(\sum_{k=1}^{a+1} X^{(k)} + \sum_{k=1}^b Y^{(k)} \right) \\ Y \stackrel{\mathcal{L}}{=} U^m \left(\sum_{k=1}^c X^{(k)} + \sum_{k=1}^{d+1} Y^{(k)} \right) \end{cases} \quad U \stackrel{\mathcal{L}}{=} \text{Unif}[0, 1]$$

Fixed point equations

$$\left\{ \begin{array}{l} X \stackrel{\mathcal{L}}{=} U^m \left(\sum_{k=1}^{a+1} X^{(k)} + \sum_{k=1}^b Y^{(k)} \right) \\ Y \stackrel{\mathcal{L}}{=} U^m \left(\sum_{k=1}^c X^{(k)} + \sum_{k=1}^{d+1} Y^{(k)} \right) \end{array} \right\} \left\{ \begin{array}{l} X \stackrel{\mathcal{L}}{=} \sum_{k=1}^{a+1} (V_k)^\sigma X^{(k)} + \sum_{k=a+2}^{S+1} (V_k)^\sigma Y^{(k)} \\ Y \stackrel{\mathcal{L}}{=} \sum_{k=1}^c (V_k)^\sigma X^{(k)} + \sum_{k=c+1}^{S+1} (V_k)^\sigma Y^{(k)} \end{array} \right.$$

Continuous time

discrete time

connection: $W^{CT} = \xi^\sigma W^{DT}$

Moments

of W^{CT} solution of

$$\left\{ \begin{array}{l} X \stackrel{\mathcal{L}}{=} U^m \left(\sum_{k=1}^{a+1} X^{(k)} + \sum_{k=1}^b Y^{(k)} \right) \\ Y \stackrel{\mathcal{L}}{=} U^m \left(\sum_{k=1}^c X^{(k)} + \sum_{k=1}^{d+1} Y^{(k)} \right) \end{array} \right.$$

Proposition *The radius of convergence of the Laplace series*

$$\sum_{p \geq 1} \frac{\mathbb{E}X^p}{p!} T^p \text{ is zero} \dots$$

Moments

of W^{CT} solution of

$$\left\{ \begin{array}{l} X \stackrel{\mathcal{L}}{=} U^m \left(\sum_{k=1}^{a+1} X^{(k)} + \sum_{k=1}^b Y^{(k)} \right) \\ Y \stackrel{\mathcal{L}}{=} U^m \left(\sum_{k=1}^c X^{(k)} + \sum_{k=1}^{d+1} Y^{(k)} \right) \end{array} \right.$$

Proposition *The radius of convergence of the Laplace series*

$$\sum_{p \geq 1} \frac{\mathbb{E} X^p}{p!} T^p \text{ is zero} \dots$$

$$\forall A > 0,$$

$$A^p \leq \frac{\mathbb{E} |X|^p}{p!}$$

Moments

of W^{CT} solution of

$$\left\{ \begin{array}{l} X \stackrel{\mathcal{L}}{=} U^m \left(\sum_{k=1}^{a+1} X^{(k)} + \sum_{k=1}^b Y^{(k)} \right) \\ Y \stackrel{\mathcal{L}}{=} U^m \left(\sum_{k=1}^c X^{(k)} + \sum_{k=1}^{d+1} Y^{(k)} \right) \end{array} \right.$$

$$A^p \leq \frac{\mathbb{E}|X|^p}{p!} \leq C^p (\log p)^p$$

$$(\mathbb{E}|X|^p)^{-\frac{1}{p}} \geq \frac{(p!)^{-\frac{1}{p}}}{C \log p} \sim \frac{C'}{p \log p}$$

Carleman's criterion:

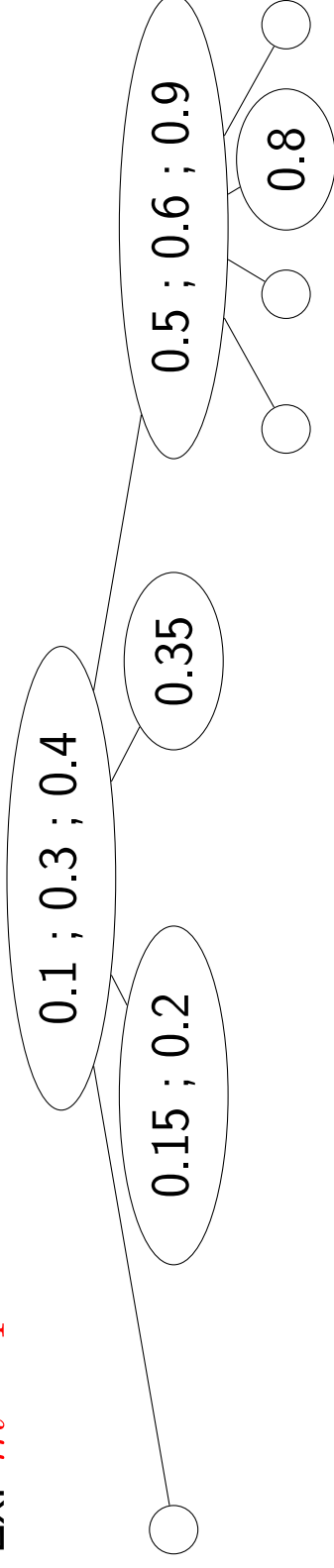
the law of X is determined by its moments.

Part III

m-ary search trees - Discrete time

m branches, at most $(m - 1)$ keys in a node.

Ex: $m = 4$



$$X_n^{DT} := \left(\begin{array}{l} \# \text{ empty nodes at time } n \\ \# \text{ nodes with 1 key at } n \\ \vdots \\ \# \text{ nodes with } (m - 2) \text{ keys at } n \end{array} \right)$$

Asymptotics of X_n^{DT}

$m \geq 27$

$$X_n^{DT} = nv_1 + \Re(n^{\lambda_2} W^{DT} v_2) + o(n^{\sigma_2}) \quad a.s.,$$

where

- $\lambda_2 = \sigma_2 + i\tau_2$ is the root of the characteristic equation

$$\prod_{k=1}^{m-1} (\lambda + k) = m!$$

having the second largest real part σ_2 and a positive imaginary part τ_2 ,

- v_1 and v_2 are two deterministic vectors,
- W^{DT} is the limit of a complex-valued martingale that admits moments of all positive orders.

“Divide and conquer” (Fill-Kapur): W^{DT} is a solution of

$$Z \stackrel{\mathcal{L}}{=} \sum_{k=1}^m (V_k)^{\lambda_2} Z^{(k)},$$

where

$(V_1, \dots, V_m) \stackrel{\mathcal{L}}{=} \text{Dirichlet}(1, \dots, 1)$:

$$\left\{ \sum_{k=1}^m V_k = 1 \right.$$

$V_k \stackrel{\mathcal{L}}{=} \text{Beta}(1, m - 1)$

Part IV

m-ary search trees - Continuous time

Continuous-time Markov process $(X^{CT}(t))_{t \geq 0}$

$\mathcal{E}xp(i)$ clock on a particle of type i ; independence of the clocks.

Branching

- if a particle i rings, replace it by 1 particle $i+1$, $i = 1, 2, \dots, m-2$.
- if a particle $m-1$ rings, replace it by m particles of type 1.

$0 < \tau_1 < \dots < \tau_n < \dots$ are the jump times

$$(X^{CT}(\tau_n))_{n \geq 0} \stackrel{\mathcal{L}}{=} (X^{DT}(n))_{n \geq 0}$$

Benefit: **independence**

$$\begin{pmatrix} \# \text{ empty nodes at time } t \\ \# \text{ nodes with 1 key at } t \\ \vdots \\ \# \text{ nodes with } (m - 2) \text{ keys at } t \end{pmatrix}$$

Asymptotics of $X^{CT}(t) :=$

$m \geq 27$

$$X(t) = e^t \xi v_1 (1 + o(1)) + 2\Re(e^{\lambda_2 t} W^{CT} v_2) (1 + o(1)) + o(e^{\lambda_2 t}),$$

where

- ξ is a positive Gamma-distributed random variable.
- W^{CT} is \mathbb{C} -valued
- $o(1)$ are a.s. and in L^p .

branching property: W^{CT} is a solution of

$$Z \stackrel{\mathcal{L}}{=} V^{\lambda_2} (Z^{(1)} + \dots + Z^{(m)}),$$

where $V \stackrel{\mathcal{L}}{=} \text{Beta}(1, m - 1)$.

Fixed point equations

$$\left\{ \begin{array}{l} X \stackrel{\mathcal{L}}{=} U^m \left(\sum_{k=1}^{a+1} X^{(k)} + \sum_{k=1}^b Y^{(k)} \right) \\ Y \stackrel{\mathcal{L}}{=} U^m \left(\sum_{k=1}^c X^{(k)} + \sum_{k=1}^{d+1} Y^{(k)} \right) \end{array} \right\} \left\{ \begin{array}{l} X \stackrel{\mathcal{L}}{=} \sum_{k=1}^{a+1} (V_k)^\sigma X^{(k)} + \sum_{k=a+2}^{S+1} (V_k)^\sigma Y^{(k)} \\ Y \stackrel{\mathcal{L}}{=} \sum_{k=1}^c (V_k)^\sigma X^{(k)} + \sum_{k=c+1}^{S+1} (V_k)^\sigma Y^{(k)} \end{array} \right.$$

continuous time

$$W^{CT} = \xi^\sigma W^{DT}$$

discrete time

$$Z \stackrel{\mathcal{L}}{=} V^{\lambda_2} (Z^{(1)} + \dots + Z^{(m)}) \quad Z \stackrel{\mathcal{L}}{=} \sum_{k=1}^m (V_k)^{\lambda_2} Z^{(k)},$$

$$W^{CT} = \xi^{\lambda_2} W^{DT}$$

Part V

Fixed point equations

Contraction method

$$Z \stackrel{\mathcal{L}}{=} V^\lambda(Z^{(1)} + \dots + Z^{(m)})$$

- find a good space:
 $\mathcal{M}_2(C)$ the space of probability distributions on \mathbb{C} admitting a second absolute moment and having C as expectation.
- find a good distance: d_2 (Wasserstein)
- define
$$K : \mathcal{M}_2(C) \rightarrow \mathcal{M}_2(C)$$
$$\mathcal{L}(Z) \mapsto \mathcal{L}(V^\lambda(Z^{(1)} + \dots + Z^{(m)})).$$

$$\prod_{k=1}^{m-1} (\lambda + k) = m!$$

- $\Re\lambda_2 > \frac{1}{2} \implies K$ is a contraction.

$$Z \stackrel{\mathcal{L}}{=} V^\lambda(Z^{(1)} + \dots + Z^{(m)})$$

Theorem

- *The support of Z is the whole complex plane \mathbb{C} ;*
- *The distribution of Z has a density with respect to the Lebesgue measure on \mathbb{C} ;*
- *Exponential moments of Z exist in a neighborhood of 0.*

Questions

- splitting, fixed point equations and contraction method for any urn with d colors.
- algebra of the distributions W depending on a, b, c, d ?
- Any idea about a “large” urn with an algorithmic meaning?