

# Combinatorics of type B permutation tableaux

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# Introduction

## Definition

A *permutation tableau* is a Young diagram filled with 0's and 1's, such that:

- ▶ There is at least a 1 per column,

- ▶ The pattern  $\begin{matrix} & & & 1 \\ & & & \vdots \\ & & & 0 \\ 1 & \dots & & \end{matrix}$  is forbidden.

## Example

1	0	0	1
1	1	0	1
0	0	1	
1	1		

The number of PTs with  $k + 1$  rows and  $n - k$  columns is the Eulerian number  $E_{n,k} = \#\{\sigma \in S_n : \text{des}(\sigma) = k\}$ .

Counting the number of *superfluous* 1's (all except the topmost of each column) gives a  $q$ -analog  $E_{n,k}(q)$  [Williams].

# Introduction

Type  $B$  permutation tableaux: defined by Lam and Williams (in relation with geometric objects such as orthogonal grassmannian...)

These are (roughly) conjugate-symmetric permutation tableaux, and are in bijection with signed permutations.

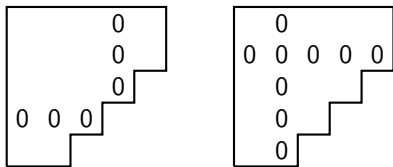
The goal of this work is to show how a good  $q$ -analog of type  $B$  Eulerian numbers arise from this definition.

## Type B permutation tableaux

Remark: A conjugate-symmetric permutation tableau contains no zero-row.

### Definition

A type B permutation tableau is obtained from a conjugate-symmetric permutation tableau by adding some zero-rows and zero-columns the following way:

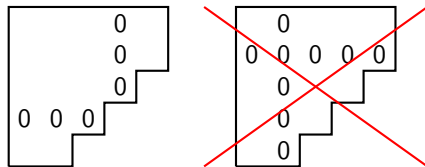


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1	1	1	0	
0	0	0		
0	1			

OK

0				
0	0	0	0	0
0				
0				
0				

## The zig-zag bijection

We use a bijection of [Steingrímsson Williams]. Label the boundary of the permutation tableau with integers for  $-n$  to  $n$ . The image of  $i$  is obtained by taking a zig-zag path, the direction East or South changing at each 1.

### Example

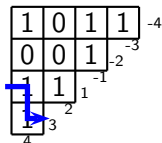
1	0	1	1	<sup>-4</sup>
0	0	1		<sup>-3</sup>
1	1			<sup>-1</sup>
1				<sup>1</sup>
				<sup>2</sup>
				<sup>3</sup>
				<sup>4</sup>

$$\pi = 3, 1, 4, -2.$$

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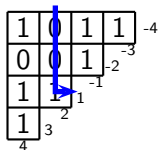
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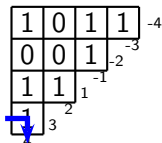
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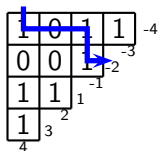


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# Crossings for signed permutations

## Definition

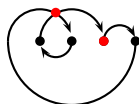
A *crossing* of a signed permutation is a pair  $(i, j) \in [n]^2$  such that

- ▶ either  $i < j \leq \pi(i) < \pi(j)$ ,
- ▶ or  $i > j > \pi(i) > \pi(j)$ ,
- ▶ or  $-i < j \leq -\pi(i) < \pi(j)$ .

This generalizes a known definition for (unsigned) permutations. We use an arrow notation such that this corresponds to proper intersection between arrows, or the limit case of two arrows 

## Example

$$\pi = 3, 1, 4, -2.$$



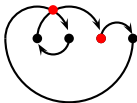
## Theorem

Via the zig-zag bijection,

- ▶ the number of superfluous 1's in type  $B$  permutation tableaux is the number of crossings in signed permutations,
- ▶  $i > 0$  is such that  $\pi(i) \geq i$  iff  $i$  label a vertical step in the South-East boundary of the permutation tableau,
- ▶ the number of  $i > 0$  with  $\pi(i) < 0$  is the number of 1's in the diagonal of the permutation tableau.

## Example

1	0	1	1	<sup>-4</sup>
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1	<sup>2</sup>	<sup>1</sup>		
	<sup>3</sup>			
	<sup>4</sup>			



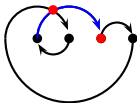
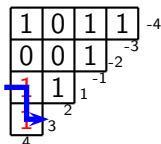
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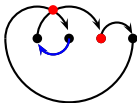
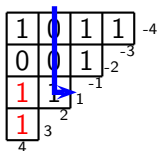
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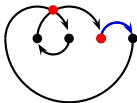
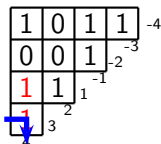
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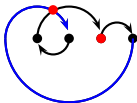
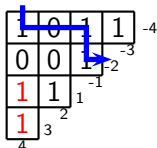


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## Eulerian numbers of type $B$

In this context, it is interesting to use:

$$\begin{aligned} \text{fdes}(\pi) &= \text{des}(\pi(-n), \dots, \pi(-1), \pi(1), \dots, \pi(n)). \\ &= 2 \times \text{des}(\pi(1), \dots, \pi(n)) + \chi[\pi(1) < 0] \end{aligned}$$

$$\text{fwex}(\pi) = 2 \times \#\{i \mid \pi(i) \geq i\} + \text{neg}(\pi)$$

(introduced by [Adin, Brenti, Roichman] and [Foata-Han]).

$\text{fdes} + 1$  and  $\text{fwex}$  are equidistributed.

We have  $\lfloor \text{fdes}(\pi)/2 \rfloor = \text{des}_B(\pi) = \text{des}(0, \pi(1), \dots, \pi(n))$ .

Similar to the counting rows and 1's in permutation tableaux, from the type B permutation tableaux we are lead to define

$$B_{n,k}(t, q) = \sum_{\pi \text{ with } \text{fwex}(\pi)=k} t^{\text{neg}(\pi)} q^{\text{cr}(\pi)},$$

### Theorem

$$B_{n,k}(1, q) = B_{n,2n+1-k}(1, q).$$

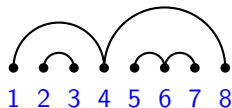
$E_{n,k}^b(q) = B_{n,2k}(1, q) + B_{n,2k+1}(1, q)$  is a  $q$ -analog of the type B Eulerian numbers such that  $E_{n,k}^b(q) = E_{n,n-k}^b(q)$ .

# Non-crossing partitions

A set partition is non-crossing if there are no  $i < j < k < \ell$  with  $i, j$  in a same block,  $k, \ell$  in another block.

There is a bijection between non-crossing permutations and non-crossing partitions given by the cycle decomposition.

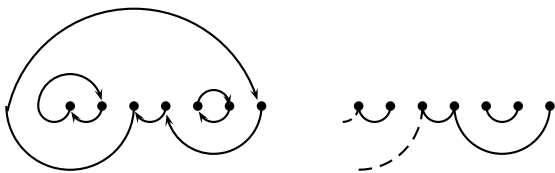
$$\pi = \{\{1, 4, 8\}, \{2, 3\}, \{5, 6, 7\}\}$$



Similarly, non-crossing signed permutations are in bijection with *non-crossing partitions of type B*.

Non-crossing partitions of classical types are defined as a sublattice of a Coxeter group. Combinatorial description in type  $B$ : a type  $B$  non-crossing partition is a couple a (type  $A$ ) non-crossing partition, and a subset of the non-nested blocks.

There is a bijection with signed permutations having no crossing, for example with  $\pi = -2, 1, -7, 3, 6, 5, 4$ :



We have  $B_{n,k}(0) = \binom{n}{k}^2$ , the Narayana number of type  $B$ .

Let  $B_n(y, t, q) = \sum_{i=1}^{2k} y^k B_{n,k}(t, q)$ .

### Theorem

(Matrix Ansatz) Let  $D$  and  $E$  be matrices,  $\langle W|$  a row vector, and  $|V\rangle$  a column vector, such that:

$$DE = qED + D + E, \quad D|V\rangle = |V\rangle, \quad \langle W|E = yt\langle W|D. \quad (1)$$

Then we have:

$$B_n(y, t, q) = \langle W|(y^2D + E)^n|V\rangle.$$

A first consequence: let  $D_q$  be the  $q$ -derivative wrt  $t$ , such that  $D_q(t^n) = [n]_q t^{n-1}$  where  $[n]_q = \frac{1-q^n}{1-q}$ . Then  $B_0(y, t, q) = 1$  and

$$B_{n+1}(y, t, q) = (y + t)D_q[(1 + yt)B_n(y, t, q)].$$

$y^2D + E$  is a tridiagonal matrix, so we can see it as a transfer matrix for weighted Motzkin paths.

So there is a continued fraction:

### Theorem

Let  $\gamma_h = y^2[h+1]_q + [h]_q + tq^h([h]_q + [h+1]_q)$  and  $\lambda_h = y[h]_q^2(y + tq^{h-1})(1 + ytq^h)$ . Then we have

$$\sum_{n \geq 0} B_n(y, t, q)z^n = \frac{1}{1 - \gamma_0 z - \frac{\lambda_1 z^2}{1 - \gamma_1 z - \frac{\lambda_2 z^2}{1 - \gamma_2 z - \dots}}}$$

From this continued fraction, we can extract:

$$B_n(y, 1, q) = \frac{1}{(1-q)^n} \sum_{k=0}^n \left( \sum_{i=0}^{2n-2k} y^i \binom{n}{k + \lceil i/2 \rceil} \binom{n}{\lfloor i/2 \rfloor} \right) \left( \sum_{j=0}^{2k} y^{j+1} q^{(j+1)(2k-j)/2} \right)$$

thanks  
for your  
attention