Combinatorics of type B permutation tableaux

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Definition

A permutation tableau is a Young diagram filled with 0’s and 1’s, such that:

- There is at least a 1 per column,

  1

- The pattern

  1 \ldots 0

  is forbidden.

The number of PTs with $k + 1$ rows and $n - k$ columns is the Eulerian number $E_{n,k} = \#\{\sigma \in S_n : \text{des}(\sigma) = k\}$.

Counting the number of superfluous 1’s (all except the topmost of each column) gives a $q$-analog $E_{n,k}(q)$ [Williams].
Type $B$ permutation tableaux: defined by Lam and Williams (in relation with geometric objects such as orthogonal grassmannian...)

These are (roughly) conjugate-symmetric permutation tableaux, and are in bijection with signed permutations.

The goal of this work is to show how a good $q$-analog of type B Eulerian numbers arise from this definition.
Type B permutation tableaux

Remark: A conjugate-symmetric permutation tableau contains no zero-row.

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![Diagram](image-url)
Type B permutation tableaux

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Definition
A type B permutation tableau is obtained from a conjugate-symmetric permutation tableau by adding some zero-rows and zero-columns the following way:

\[
\begin{array}{cccccc}
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 \\
0 & 0 & 0 \\
0 & 1 \\
\end{array}
\]

OK
The zig-zag bijection

We use a bijection of [Steingrímsson Williams]. Label the boundary of the permutation tableau with integers for $-n$ to $n$. The image of $i$ is obtained by taking a zig-zag path, the direction East or South changing at each 1.

Example

```
1 0 1 1
0 0 1
1 1
1 3
```

$\pi = 3, 1, 4, -2.$
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Example

\[
\begin{array}{cccccc}
1 & 0 & 1 & 1 & -4 \\
0 & 0 & 1 & -3 & 1 \\
1 & 1 & -2 & -1 & 4 \\
1 & 3 & -4 & 2 & -2 \\
\end{array}
\]

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Example

\begin{center}
\begin{array}{cccc}
1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -2 \\ 1 & 1 & -1 & 2 \\ 1 & 3 & 2 & -1 \\
\end{array}
\end{center}

$\pi = 3, 1, 4, -2$. 
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Example

\[
\begin{array}{cccc}
1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 3 & 2 & 1 \\
\end{array}
\quad \pi = 3, 1, 4, -2.
\]
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Example

\[
\begin{array}{cccc}
& 0 & 1 & 1 \\
1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 3 & 2 & 1 \\
\end{array}
\]

\(\pi = 3, 1, 4, -2\).
Crossings for signed permutations

Definition
A crossing of a signed permutation is a pair \((i, j) \in [n]^2\) such that
- either \(i < j \leq \pi(i) < \pi(j)\),
- or \(i > j > \pi(i) > \pi(j)\),
- or \(-i < j \leq -\pi(i) < \pi(j)\).

This generalizes a known definition for (unsigned) permutations. We use an arrow notation such that this corresponds to proper intersection between arrows, or the limit case of two arrows \(\nearrow\searrow\)

Example
\[
\pi = 3, 1, 4, -2.
\]
Theorem
Via the zig-zag bijection,

- the number of superfluous 1's in type B permutation tableaux is the number of crossings in signed permutations,
- $i > 0$ is such that $\pi(i) \geq i$ iff $i$ label a vertical step in the South-East boundary of the permutation tableau,
- the number of $i > 0$ with $\pi(i) < 0$ is the number of 1's in the diagonal of the permutation tableau.

Example

\[
\begin{array}{cccc}
1 & 0 & 1 & 1 \\
0 & 0 & 1 & \square \\
1 & 1 & 1 & \square \\
1 & 3 & 1 & \square \\
\end{array}
\]

$\pi = 3, 1, 4, -2.$
Theorem
Via the zig-zag bijection,

- **the number of superfluous 1's in type B permutation tableaux is the number of crossings in signed permutations,**
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- **the number of $i > 0$ with $\pi(i) < 0$ is the number of 1's in the diagonal of the permutation tableau.**

Example

\[
\begin{array}{cccc}
1 & 0 & 1 & 1 \\
0 & 0 & 1 & -2 \\
1 & 1 & 1 & -3 \\
4 & -4 & -1 & -2 \\
\end{array}
\]

$\pi = 3, 1, 4, -2$. 
Theorem
Via the zig-zag bijection,

- the number of superfluous 1's in type B permutation tableaux is the number of crossings in signed permutations,
- \( i > 0 \) is such that \( \pi(i) \geq i \) iff \( i \) label a vertical step in the South-East boundary of the permutation tableau,
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Example
\[
\begin{array}{cccc}
1 & 0 & 1 & 1 \\
0 & 0 & 1 & -4 \\
1 & 1 & -1 & -2 \\
1 & 1 & -1 & -3 \\
4 & 3 & -2 & -1 \\
\end{array}
\]
\( \pi = 3, 1, 4, -2. \)
Theorem

Via the zig-zag bijection,

- the number of superfluous 1's in type B permutation tableaux is the number of crossings in signed permutations,
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Example

\[
\begin{array}{cccc}
1 & 0 & 1 & 1 \\
0 & 0 & 1 & -3 \\
1 & 1 & 1 & -2 \\
1 & 1 & 1 & -4 \\
\end{array}
\]

$\pi = 3, 1, 4, -2.$
Theorem

Via the zig-zag bijection,

- the number of superfluous 1's in type B permutation tableaux is the number of crossings in signed permutations,
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Example

$$
\begin{array}{cccc}
1 & 0 & 1 & 1 \\
0 & 0 & 1 & -4 \\
1 & 1 & 1 & -3 \\
1 & 3 & 2 & -1 \\
4 & & & \\
\end{array}
$$

$\pi = 3, 1, 4, -2.$
Eulerian numbers of type $B$

In this context, it is interesting to use:

$$f_{\text{des}}(\pi) = \text{des}(\pi(-n), \ldots, \pi(-1), \pi(1), \ldots, \pi(n)).$$

$$= 2 \times \text{des}(\pi(1), \ldots, \pi(n)) + \chi[\pi(1) < 0]$$

$$f_{\text{fwex}}(\pi) = 2 \times \#\{i \mid \pi(i) \geq i\} + \text{neg}(\pi)$$

(introduced by [Adin, Brenti, Roichman] and [Foata-Han]).

$f_{\text{des}} + 1$ and $f_{\text{fwex}}$ are equidistributed.

We have $\lfloor f_{\text{des}}(\pi)/2 \rfloor = \text{des}_B(\pi) = \text{des}(0, \pi(1), \ldots, \pi(n))$. 


Similar to the counting rows and 1’s in permutation tableaux, from the type B permutation tableaux we are lead to define

\[ B_{n,k}(t, q) = \sum_{\pi \text{ with } \text{fwex}(\pi) = k} t^{\text{neg}(\pi)} q^{\text{cr}(\pi)}, \]

**Theorem**

\( B_{n,k}(1, q) = B_{n,2n+1-k}(1, q). \)

\( E_{n,k}^b(q) = B_{n,2k}(1, q) + B_{n,2k+1}(1, q) \) is a \( q \)-analog of the type B Eulerian numbers such that \( E_{n,k}^b(q) = E_{n,n-k}^b(q). \)
Non-crossing partitions

A set partition is non-crossing if there are no $i < j < k < \ell$ with $i, j$ in a same block, $k, \ell$ a one other block.

There is a bijection between non-crossing permutations and non-crossing partitions given by the cycle decomposition.

\[ \pi = \{\{1, 4, 8\}, \{2, 3\}, \{5, 6, 7\}\} \]

Similarly, non-crossing signed permutations are in bijection with non-crossing partitions of type $B$. 
Non-crossing partitions of classical types are defined as a sublattice of a Coxeter group. Combinatorial description in type $B$: a type $B$ non-crossing partition is a couple a (type $A$) non-crossing partition, and a subset of the non-nested blocks.

There is a bijection with signed permutations having no crossing, for example with $\pi = -2, 1, -7, 3, 6, 5, 4$:

![Diagram showing non-crossing partitions]

We have $B_{n,k}(0) = \binom{n}{k}^2$, the Narayana number of type $B$. 
Let \( B_n(y, t, q) = \sum_{i=1}^{2k} y^k B_{n,k}(t, q) \).

**Theorem**

*(Matrix Ansatz)* Let \( D \) and \( E \) be matrices, \( \langle W \mid \text{a row vector, and} \mid V \rangle \text{a column vector, such that:}*

\[
DE = qED + D + E, \quad D\mid V \rangle = \mid V \rangle, \quad \langle W \mid E = yt\langle W \mid D. \quad (1)
\]

Then we have:

\[
B_n(y, t, q) = \langle W \mid (y^2 D + E)^n \mid V \rangle.
\]

A first consequence: let \( D_q \) be the \( q \)-derivative wrt \( t \), such that \( D_q(t^n) = [n]_q t^{n-1} \) where \( [n]_q = \frac{1-q^n}{1-q} \). Then \( B_0(y, t, q) = 1 \) and

\[
B_{n+1}(y, t, q) = (y + t)D_q[(1 + yt)B_n(y, t, q)].
\]
$y^2D + E$ is a tridiagonal matrix, so we can see it as a transfer matrix for weighted Motzkin paths.

So there is a continued fraction:

**Theorem**

Let $\gamma_h = y^2[h + 1]_q + [h]_q + tyq^h([h]_q + [h + 1]_q)$ and $\lambda_h = y[h]_q^2(y + tq^{h-1})(1 + ytq^h)$. Then we have

$$\sum_{n \geq 0} B_n(y, t, q)z^n = \frac{1}{1 - \gamma_0z - \frac{\lambda_1z^2}{1 - \gamma_1z - \frac{\lambda_2z^2}{1 - \gamma_2z - \ldots}}}.$$ 

From this continued fraction, we can extract:

$$B_n(y, 1, q) = \frac{1}{(1 - q)^n} \sum_{k=0}^{n} \left( \sum_{i=0}^{2n-2k} y^i (k+\lfloor i/2 \rfloor) \binom{n}{\lfloor i/2 \rfloor} \right) \left( \sum_{j=0}^{2k} y^{j+1} q^{(j+1)(2k-j)/2} \right).$$
thanks
for your
attention