

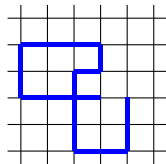
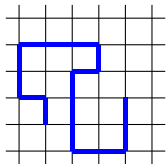
Fractal Self-Avoiding Walks

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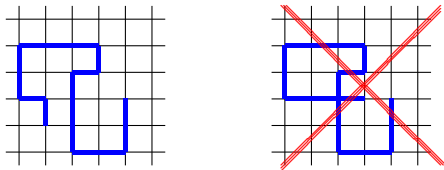
A Basic Definition

A **Self Avoiding Walk (SAW)** on the square lattice $Z \times Z$ is a walk which does not pass twice through the same vertex.



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The same definition holds for other lattices, higher dimensions, ...

The General Problem

Questions :

- ▶ What is the number c_n of SAWs of length n ?
- ▶ What is the average distance d_n between the two extremities of a SAW of length n ?

Conjectures :

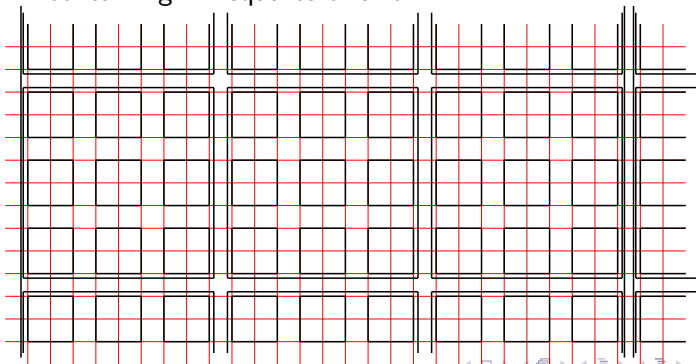
- ▶ $c_n \approx \mu^n n^{\gamma-1}$, where $\mu = \sqrt{\frac{7+\sqrt{30261}}{26}}$
- ▶ $d_n \approx n^\nu$, where $\nu = 3/4$

What we can do :

- ▶ Define families of SAWs on which μ or ν can be calculated or approximated.

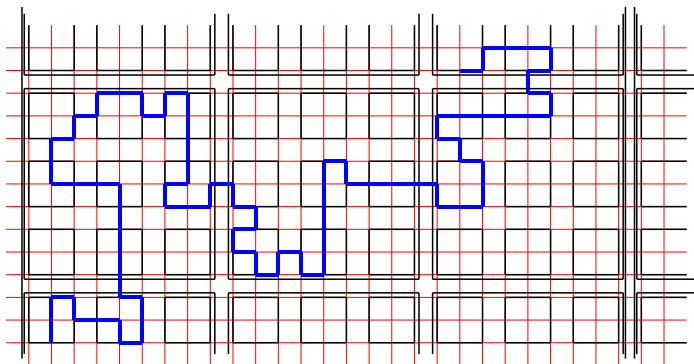
Fractal Self-Avoiding Walks : Definition

K an odd integer > 1 . We simultaneously partition the square lattice into squares of size K^p (for every $p > 0$), each square of size K^{p+1} containing K^2 squares of size K^p



A SAW is *K-Fractal* (F_K) if, for any square, of any size :

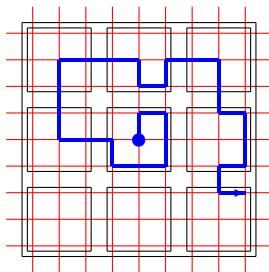
- ▶ It does not pass twice through the same square.
- ▶ When it leaves a square, it is through the middle of a side.



We denote by \mathcal{F}_K the set of all these SAWs.

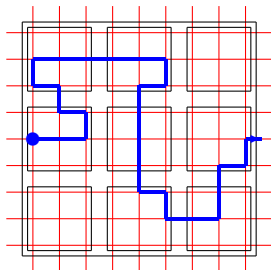
Fractal Self-Avoiding Walks : Definition

- ▶ *centered* : There exists $p > 0$ and a square of size K^p such that the walk begins at the center of the square and stays inside this square.

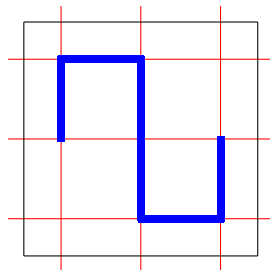


Property : $\lim_{K \rightarrow \infty} \mathcal{F}_K$ is the set of all SAWs

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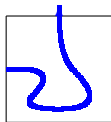
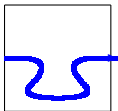


\approx



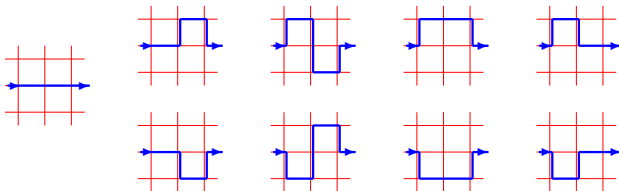
Counting Fractal Self-Avoiding Walks

For Directed F_K SAWs :

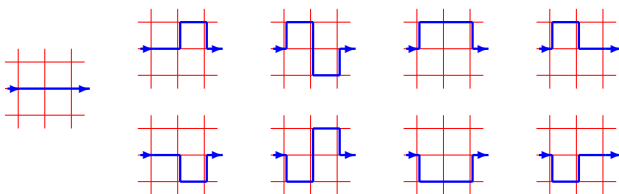


Counting Fractal Self-Avoiding Walks : a Naïve Method

$$K = 3$$



$$K = 3$$



$$S = s^3 + 2st^6 + 6st^4$$

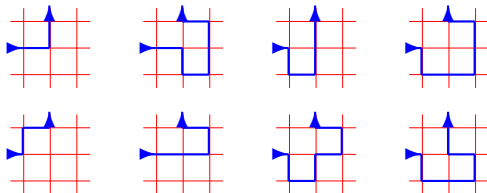
Where $s(z) = \sum_{n \in \mathbb{N}} S_k(n)z^n$, $t(z) = \sum_{n \in \mathbb{N}} T_k(n)z^n$ and

$$S(z) = \sum_{n \in \mathbb{N}} S_{k+1}(n)z^n.$$

$S_k(n)$ (resp. $T_k(n)$) is the number of 3-fractal SAWs which cross (resp turn into) a $3^k \times 3^k$ square.

Counting Fractal Self-Avoiding Walks : a Naïve Method

$$K = 3$$



$$T = t^3 + t^7 + s^2t + 2s^2t^3 + 3s^2t^5$$

$$\text{Where } T(z) = \sum_{n \in \mathbb{N}} T_{k+1}(n)z^n.$$

Counting Fractal Self-Avoiding Walks : a Naïve Method

$$K = 3 : S = s^3 + 2st^6 + 6st^4, T = t^3 + t^7 + s^2t + 2s^2t^3 + 3s^2t^5$$

Counting Fractal Self-Avoiding Walks : a Naïve Method

$$K = 3 : S = s^3 + 2st^6 + 6st^4, T = t^3 + t^7 + s^2t + 2s^2t^3 + 3s^2t^5$$

$K = 5 :$

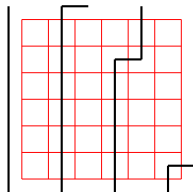
$$\begin{aligned} S = & s^5 + 4s^{13}t^{10} + 6s^{13}t^{12} + 12s^{11}t^8 + 44s^{11}t^{10} + 56s^{11}t^{12} + 18s^{11}t^{14} + \\ & 4s^9t^6 + 56s^9t^8 + 168s^9t^{10} + 198s^9t^{12} + 80s^9t^{14} + 8s^9t^{16} + 32s^7t^6 + \\ & 134s^7t^8 + 306s^7t^{10} + 372s^7t^{12} + 222s^7t^{14} + 52s^7t^{16} + 4s^7t^{18} + 20s^5t^4 + \\ & 64s^5t^6 + 174s^5t^8 + 324s^5t^{10} + 368s^5t^{12} + 256s^5t^{14} + 98s^5t^{16} + \\ & 16s^5t^{18} + 2s^5t^{20} + 20s^3t^4 + 60s^3t^6 + 124s^3t^8 + 174s^3t^{10} + 164s^3t^{12} + \\ & 122s^3t^{14} + 60s^3t^{16} + 8s^3t^{18} + 30st^8 + 32st^{10} + 8st^{12} + 8st^{14} + 6st^{16} \\ T = & 2s^{14}t^9 + 2s^{14}t^{11} + 4s^{12}t^7 + 14s^{12}t^9 + 14s^{12}t^{11} + 6s^{12}t^{13} + \\ & 16s^{10}t^7 + 46s^{10}t^9 + 76s^{10}t^{11} + 52s^{10}t^{13} + 12s^{10}t^{15} + 14s^8t^5 + 54s^8t^7 + \\ & 171s^8t^9 + 278s^8t^{11} + 228s^8t^{13} + 68s^8t^{15} + 6s^8t^{17} + 4s^6t^3 + 26s^6t^5 + \\ & 123s^6t^7 + 306s^6t^9 + 483s^6t^{11} + 406s^6t^{13} + 168s^6t^{15} + 26s^6t^{17} + s^4t + \\ & 4s^4t^3 + 22s^4t^5 + 86s^4t^7 + 239s^4t^9 + 370s^4t^{11} + 337s^4t^{13} + 194s^4t^{15} + \\ & 62s^4t^{17} + 14s^4t^{19} + 2s^4t^{21} + 4s^2t^3 + 10s^2t^5 + 19s^2t^7 + 54s^2t^9 + \\ & 85s^2t^{11} + 86s^2t^{13} + 45s^2t^{15} + 18s^2t^{17} + 6s^2t^{19} + t^5 + 3t^9 + 5t^{13} \end{aligned}$$

Naïve Enumeration : $\rightarrow K = 7$.

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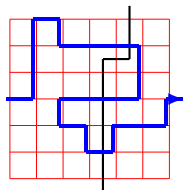
Finite Lattice Method (A.J. Guttmann & all.) : $\rightarrow K = 11$.

Counting Fractal Self-Avoiding Walks : the Finite Lattice Method



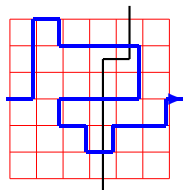
The **hooks** $\langle 0, 7 \rangle$, $\langle 3, 0 \rangle$, $\langle 5, 2 \rangle$ and $\langle 7, 6 \rangle$ inside a 7×7 square.

Counting Fractal Self-Avoiding Walks : the Finite Lattice Method

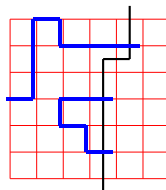


a SAW

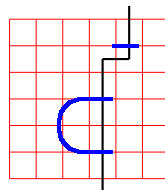
Counting Fractal Self-Avoiding Walks : the Finite Lattice Method



a SAW



a Half-SAW

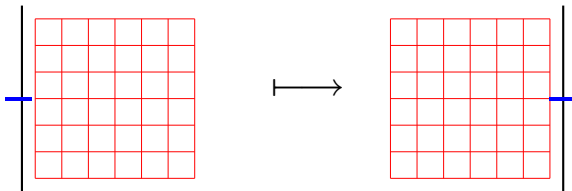


a Configuration

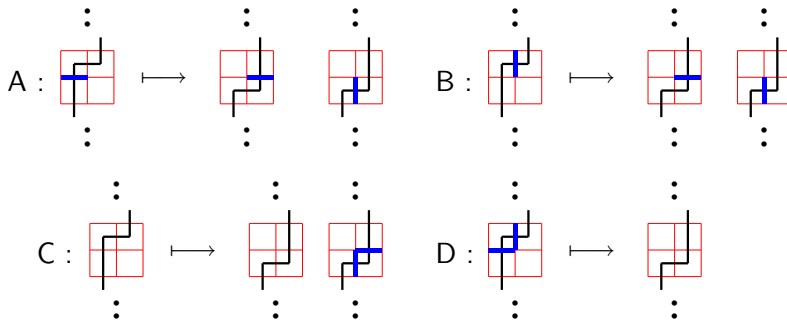
\mathcal{C} a configuration

$\alpha_{\mathcal{C}}(s, t)$ is the number of Half-SAWs corresponding to the configuration \mathcal{C} having s straights and t turns.

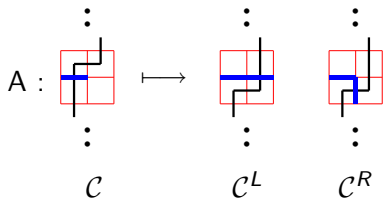
Counting Fractal Self-Avoiding Walks : the Finite Lattice Method



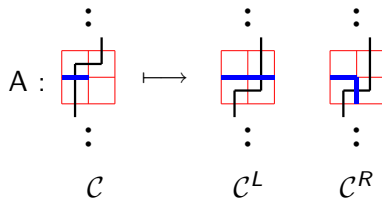
Counting Fractal Self-Avoiding Walks : the Finite Lattice Method



Counting Fractal Self-Avoiding Walks : the Finite Lattice Method



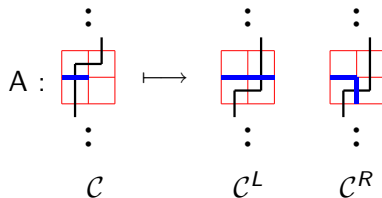
Counting Fractal Self-Avoiding Walks : the Finite Lattice Method



$$\alpha_{\mathcal{C}^L}(s+1, t) = \alpha_{\mathcal{C}}(s, t)$$

$$\alpha_{\mathcal{C}^R}(s, t+1) = \alpha_{\mathcal{C}}(s, t)$$

Counting Fractal Self-Avoiding Walks : the Finite Lattice Method



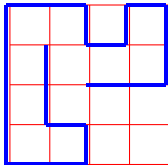
$$\alpha_{\mathcal{C}^L}(s+1, t) := \alpha_{\mathcal{C}}(s, t) + \alpha_{\mathcal{C}^L}(s+1, t)$$

$$\alpha_{\mathcal{C}^R}(s, t+1) := \alpha_{\mathcal{C}}(s, t) + \alpha_{\mathcal{C}^R}(s, t+1)$$

By dichotomy, it is possible to estimate the radius of convergence of $\lim_{k \rightarrow \infty} \sum_{n \in \mathbb{N}} S_k(n) z^n$ and $\lim_{k \rightarrow \infty} \sum_{n \in \mathbb{N}} T_k(n) z^n$. From these, we get :

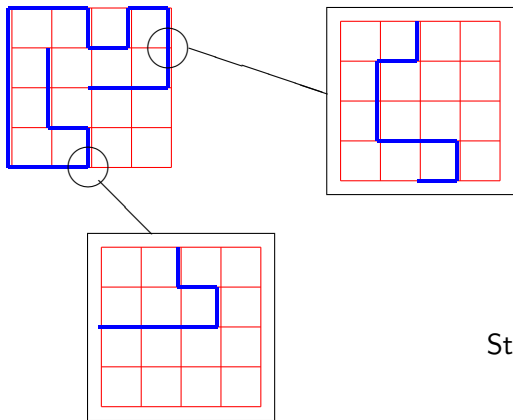
- ▶ $\mathcal{F}_3 : \mu \approx 1.758$
- ▶ $\mathcal{F}_5 : \mu \approx 2.035$
- ▶ $\mathcal{F}_7 : \mu \approx 2.180$
- ▶ $\mathcal{F}_9 : \mu \approx 2.2694173$
- ▶ $\mathcal{F}_{11} : \mu \approx 2.330$

Generating Fractal Self Avoiding Walks



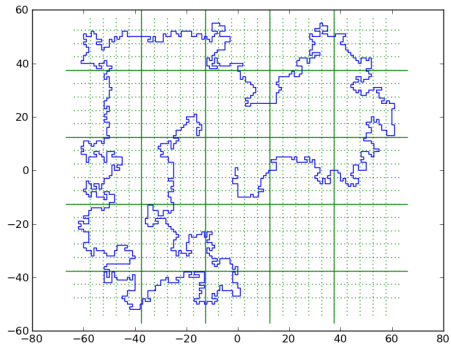
Step 0

Generating Fractal Self Avoiding Walks

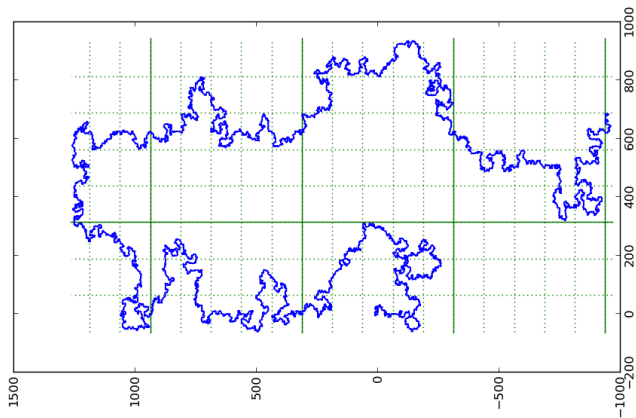


Step 1, ...

Generating Fractal Self Avoiding Walks



Generating Fractal Self Avoiding Walks



We consider the set of K -fractal SAWs generated with p steps.

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- ▶ The distance between the two extremities of a SAW is in $\approx K^p$
- ▶ The average length of a SAW is in $[l_S^p, l_T^p]$, where l_S (resp. l_T) is the average length of a SAW crossing (resp. turning into) a $K \times K$ square.
- ▶ $l_S > K$.

Property : *The critical exponent ν for the K -Fractal SAW is < 1 , i.e. the average distance between the two extremities of a K -Fractal SAW of length n is $\approx n^\nu$, with $\nu < 1$.*

Example : For $K = 3$, $\nu \in [0.64, 0.66]$.

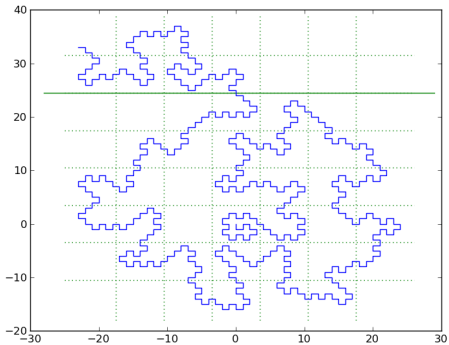
Conjecture : *This is also true for the uniform distribution.*

Variants

Sinuuous K -Fractal SAW (M. Bousquet-Mélou) :

A K -Fractal SAW is **Sinuuous** if it never cross a square from one side to the opposite one, an turns on each vertex.

Variants



$$K = 3 : T = t^3 + t^7$$

$$K = 5 : T = 3t^9 + 5t^{13} + t^5$$

$$K = 7 : T = 6t^{11} + 28t^{19} + t^7 + 34t^{23} + 15t^{15} + 23t^{27} + 5t^{31}$$

...

$$K = 3 : \mu \approx 1,210$$

$$K = 5 : \mu \approx 1,288$$

$$K = 7 : \mu \approx 1,335$$

$$K = 9 : \mu \approx 1,376$$

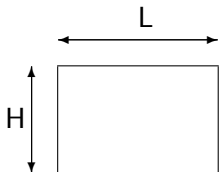
$$K = 11 : \mu \approx 1,405$$

$$K = 13 : \mu \approx 1,427$$

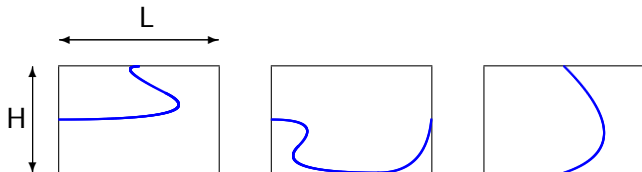
$$K = 15 : \mu \approx 1,444$$

$$K = 17 : \mu \approx 1,458$$

It is possible to take rectangles instead of squares.
 L and H two odd numbers, $1 < H \leq L$.



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 L and H two odd numbers, $1 < H \leq L$.



$$H = 5$$

$$L = 5 : \mu \approx 2.035$$

$$L = 11 : \mu \approx 2.141$$

$$L = 15 : \mu \approx 2.157$$

$$L = 21 : \mu \approx 2.168$$

$$H = 7$$

$$L = 7 : \mu \approx 2.180$$

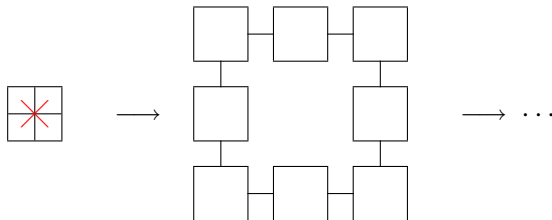
$$L = 11 : \mu \approx 2.244$$

$$L = 15 : \mu \approx 2.2690334$$

It is possible to forbid some points.

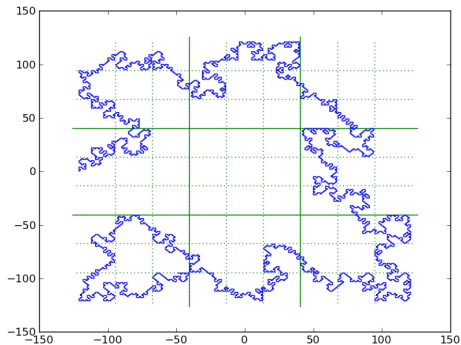


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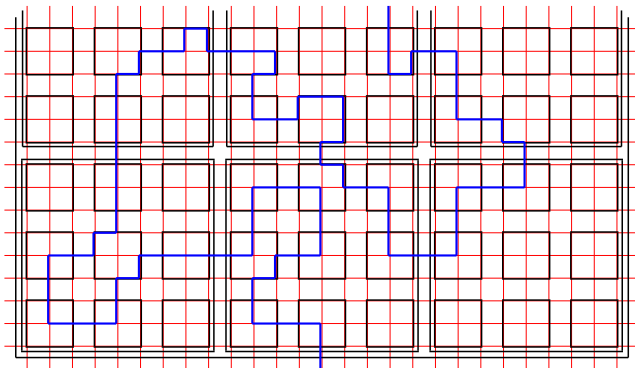
→ the Sierpinski carpet

Variants



$$\mu \approx 1.207$$

- A SAW is *K-Quasi-Fractal* (QF_K) if, for any square, of any size :
- ▶ When it leaves a square, it is through the middle of a side.



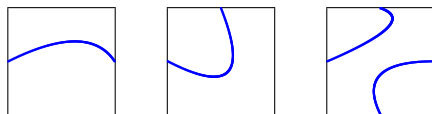
QF_K SAWs can be studied with the same tools that F_K SAWs



$K = 3$:

- ▶ $S = s^3 + 6st^4 + 4st^4c^2 + 2st^6 + 2s^3t^4c^2 + 4st^6c$
- ▶ $T = s^2t + t^3 + 3t^5c^3 + s^2t^3c + 3s^2t^5 + 2s^2t^3 + t^7 + 2s^2t^5c^2 + 2s^2t^5c + 2t^5c$
- ▶ $C = s^4c + 2s^2t^4 + 8t^4c^5 + 4s^2t^4c + 6t^6c^2 + t^6$

QF_K SAWs can be studied with the same tools that F_K SAWs



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- ▶ $C = s^4c + 2s^2t^4 + 8t^4c^5 + 4s^2t^4c + 6t^6c^2 + t^6$

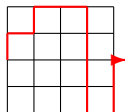
$K = 3 : \mu \approx 1.862$

$K = 5 : \mu \approx 2.155$

$K = 7 : \mu \approx 2.312$

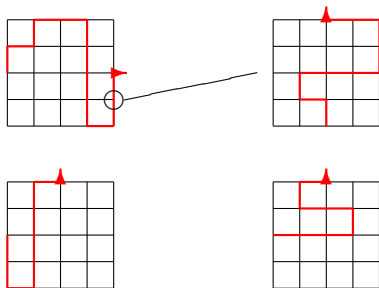
A fractal SAW is *K-Directed-Fractal* (DF_K) if, for any square, of any size :

- ▶ When it crosses a square from the West to the East, it never goes West, ...



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- ▶ When it crosses a square from the West to the East, it never goes West, ...



$$K = 43 : \mu \approx 2.325$$