Polyominoes determined by permutations

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Outline

1. Permutominoes
2. Permutominoes Enumeration
   - Column-convex permutominoes
     - A generating tree for column-convex permutominoes
     - Enumeration of column-convex permutominoes
     - Numerical analysis
   - Convex permutominoes
     - Enumeration through the ECO method
     - Bijective enumeration of some classes of convex permutominoes
3. Permutations associated with permutominoes
   - Permutations associated with convex permutominoes
   - Square permutations
   - Permutations and permutominoes
4. Further work
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A permutomino of size \( n \) is a polyomino (with no holes) having \( n \) rows and \( n \) columns, such that for each abscissa (ordinate) between 1 and \( n + 1 \) there is exactly one vertical (horizontal) bond in the boundary of \( P \) with that coordinate.
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A permutomino $P$ of size $n$ is uniquely defined by a pair of permutations of size $n + 1$, denoted by $\pi_1(P)$ and $\pi_2(P)$, called permutations associated with $P$. 
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$\pi_1 = (6, 3, 9, 8, 12, 11, 13, 1, 5, 10, 4, 7, 2)$
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\[ \pi_1 = (9, 6, 8, 13, 11, 10, 12, 3, 4, 7, 2, 5, 1) \]

\[ \pi_2 = (9, 6, 8, 13, 11, 10, 12, 3, 4, 7, 2, 5, 1) \]
A pair of permutations \((\pi_1, \pi_2)\) such that \(\pi_1(i) \neq \pi_2(i)\) may not define a permutomino.

- The boundary may be non connected;
- The boundary may cross itself.
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\[
\begin{align*}
\pi_1 &= (2, 1, 3, 4, 5, 7, 6) \\
\pi_2 &= (3, 2, 1, 5, 7, 6, 4)
\end{align*}
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\pi_1 &= (2, 4, 1, 6, 7, 3, 5) \\
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The singular locus of a Schubert variety

- Introduced in 2003 by Kassel, Lascoux, and Reutenauer, with the aim of characterizing the singular loci of the Schubert varieties $X_\mu$ in the flag variety for $GL_n(C)$.
- The name *Permutomino* (or ‘*Permutaomino*’) was coined by Incitti (2006), while studying the problem of determining the $\tilde{R}$-polynomials associated with a pair $(\mu, \nu)$ of permutations.
Permutominoes: the origins

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The graph of a permutation $\mu \in S_n$ is the set of points $(i, \mu(i))$, with $i \in [n] = \{1, \ldots, n\}$. 

**Figure:** Graph of $\mu = (3, 1, 5, 4, 7, 2, 9, 8, 6)$
Permutominoes: the origins

**Bruhat order**

For each $h, k \in [n]$ we set

$$
\mu[h, k] = |\{i \in [n] : i \leq h, \mu(i) \geq k\}|
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$\mu[6,2]=5$
Given $\mu, \nu \in S_n$, and $h, k \in [n]$ we set:

$$(\mu, \nu)[h, k] = \nu(h, k) - \mu(h, k)$$

$(\mu, i)[6, 2] = 1$
Proposition (Björner, Brenti, 2005)

Given $\mu, \nu \in S_n$, $\mu \leq \nu$ in the Bruhat order iff $(\mu, \nu)[h, k] \geq 0$, for all $(h, k) \in \mathbb{N}^2$. 
The planar representation of $(\mu, \nu)$ is obtained as follows:

- we use the graphs of $\mu$, $\nu$,
- if $\mu(i) = \nu(i)$ we have a double point,
- we connect the other points using horizontal and vertical lines.
Permutominoes: the origins

Planar representation of a pair of permutations

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Figure: Planar representation of $(\mu, \nu)$, with $\mu = (3, 1, 5, 4, 7, 2, 9, 8, 6)$, $\nu = (7, 8, 2, 4, 9, 6, 3, 1, 5)$
Planar representation of a pair of permutations

Bijective correspondence between the disjoint cycles of $\mu^{-1}\nu$ and the closed paths in the planar representation of $(\mu, \nu)$. 

![Diagram of a planar representation of a pair of permutations with closed paths indicated.]
Permutominoes: the origins

Given $\mu, \nu \in S_n$, the *multiplicity mapping* of $(\mu, \nu)$:

$$(\mu, \nu)[h, k] = \nu(h, k) - \mu(h, k), \text{ with } (h, k) \in \mathbb{R}^2$$
Permutominoes: the origins

Diagram of a pair of permutations

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$$(\mu, \nu)[h, k] = \nu(h, k) - \mu(h, k), \text{ with } (h, k) \in \mathbb{R}^2$$
A pair \((\mu, \nu)\) is a *permutomino* if

- \((\mu, \nu)[h, k] \in \{0, 1\}\), for all \((h, k) \in \mathbb{R}^2\);
- \((\mu, \nu)\) has no double points;
- the support of \((\mu, \nu)\) is connected and with no holes.
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Permutominoes: the origins

Generation of orthogonal polygons

- Permutominoes have been considered independently by Tomás, Bajuelos (2004);
- Quadratic-time algorithm for the random generation of orthogonal polygons with a fixed number of vertices;
- *Orthogonal polygons*: simple polygons without holes where the edges meet at right angles.
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Permutominoes: the origins

- Generic orthogonal polygons with $n$ vertices may be obtained from *grid $n$-ogons*: $n$-vertex orthogonal polygons having exactly one edge for every line of the grid.
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Permutominoes: the origins

Generation of orthogonal polygons

Tomás, Bajuelos (2004) provide a quadratic-time algorithm for the random generation of grid \( n \)-ogons by means of the \textit{INFLATE-PASTE} algorithm.
Contents of the talk

1. Enumeration of (various classes of) permutominoes according to the size;
2. Characterization and enumeration of pairs of permutations defining (various classes of) permutominoes.
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We consider the enumeration of some classes of permutominoes:

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- convex permutominoes,
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We present an algorithm for the recursive generation of column-convex permutominoes such that each object of size $n+1$ is obtained in a unique way from an object of size $n$.

We follow the ECO method, Enumeration of Combinatorial Objects developed by Pinzani and his team starting from '90.

ECO method has been applied to several problems of enumeration and generation of combinatorial objects.

It is strictly related to the concept of generating tree introduced by West (1994).
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To a column-convex permutomino $P$ we assign the label $(h, d, w)$. 
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(4,2,1)
Reentrant points

- Encode the boundary of the permutomino using north, east, south and west unit steps.
- Any occurrence of a sequence $NE$, $ES$, $SW$, or $WN$ in the word encoding $P$ defines a salient point of $P$; any occurrence of a sequence $EN$, $SE$, $WS$, or $NW$ defines a reentrant point of $P$. 
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![Diagram showing reentrant points](image-url)
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Reenrant points

The difference between the number of salient and reenrant points in any closed path is equal to 4. In a column-convex permutomino of size $n$ we have exactly one reentrant point for each abscissa between 1 and $n - 1$. 
Let $C_n$ be the set of column-convex permutominoes of dimension $n$.

We present a construction such that each element of $C_{n+1}$ is produced from exactly one element of $C_n$.

The idea is to let permutominoes grow from size $n$ to $n+1$ by adding a new reentrant point on the right.

Our construction consists of four operations, $\alpha, \beta, \gamma, \delta$:

- $\alpha$ adds an EN-reentrant point;
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Operation $\alpha$

$$(h, d, w) \sim (h - i + 1, d + i, w) \quad 1 \leq i \leq h + 1$$
Operation $\alpha$

$$(h, d, w) \leadsto (h - i + 1, d + i, w) \quad 1 \leq i \leq h + 1$$
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Polyominoes determined by permutations

Permutominoes Enumeration

Column-convex permutominoes

Operation $\beta$

\[(h, d, w) \sim (d + h - i + 1, i, w) \quad 1 \leq i \leq d\]
Polyominoes determined by permutations

— Permutominoes Enumeration
— Column-convex permutominoes

Operation $\beta$

$$(h, d, w) \leadsto (d + h - i + 1, i, w) \quad 1 \leq i \leq d$$
Operation $\gamma$

$$(h, d, w) \sim (h, i, d + w - i + 1) \quad 1 \leq i \leq d$$
Operation $\gamma$

$$(h, d, w) \leadsto (h, i, d + w - i + 1) \quad 1 \leq i \leq d$$
Operation $\delta$

$(h, d, w) \rightsquigarrow (h, d + i, w - i + 1) \quad 1 \leq i \leq w + 1$
Polyominoes determined by permutations

Permutominoes Enumeration

Column-convex permutominoes

Operation $\delta$

\[(h, d, w) \leadsto (h, d + i, w - i + 1) \quad 1 \leq i \leq w + 1\]
The generating operator

For a column-convex permutomino $P$ we denote $\vartheta(P)$ the set of permutominoes produced by $P$ through the application of $\alpha$, $\beta$, $\gamma$, and $\delta$.

Theorem (Beaton, Disanto, Guttmann, R, 2010)

Any column-convex permutomino of size $n \geq 2$ is uniquely obtained through the application of the operator $\vartheta$ to a column-convex permutomino of size $n - 1$. 

The recursive construction of $\vartheta$ can be formally described using 
generating trees (or succession rules). The application of $\vartheta$ to a 
column-convex permutomino with a generic label $(h, d, w)$, produces 
$2d + h + w + 2$ column-convex permutominoes according to the 
following rule:

\[
(h, d, w) \overset{\alpha}{\rightsquigarrow} (h - i + 1, d + i, w) \quad 1 \leq i \leq h + 1 \\
\overset{\beta}{\rightsquigarrow} (d + h - i + 1, i, w) \quad 1 \leq i \leq d \\
\overset{\gamma}{\rightsquigarrow} (h, i, d + w - i + 1) \quad 1 \leq i \leq d \\
\overset{\delta}{\rightsquigarrow} (h, d + i, w - i + 1) \quad 1 \leq i \leq w + 1.
\]

with root $(0, 1, 0)$, which is the label of the one cell permutomino.
The functional equation

The construction of operator $\vartheta$ can be translated onto generating functions:

$$F(x, y, z) = \sum_{P \in C} x^{h(P)} y^{d(P)} z^{w(P)}$$

$$= y + xy + yz + 2y^2 + 3x^2y + 2xyz + 3z^2y + 4xy^2 + 4zy^2 + 6y^3 + \ldots$$
The functional equation

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$$F(x, y, z) = y + \sum_{P \in \mathcal{C}} x^h y^{d+1} z^w + \ldots + x^0 y^{d+h+1} z^w + \sum_{P \in \mathcal{C}} x^{d+h} y^1 z^w + \ldots + x^{h+1} y^d z^w$$

$$+ \sum_{P \in \mathcal{C}} x^h yz^{d+w} + \ldots + x^h y^d z^{w+1} + \sum_{P \in \mathcal{C}} x^h y^{d+1} z^w + \ldots + x^h y^{d+w+1} z^0,$$
The functional equation

We obtain the following functional equation

$$F(x, y, z) = y + \frac{yz}{z - y} F(x, z, z) - \frac{y^2}{z - y} F(x, y, y) + \frac{xy}{x - y} F(x, x, z) - \frac{y^2}{x - y} F(y, y, z).$$

From this equation we are able to compute the first 200 terms of sequence $f_n$ beginning with:

$$1, 4, 22, 152, 1262, 12232, 135544, 1690080, 23417928, \ldots$$

We have not been able to solve the equation, just to find lower and upper bounds for the sequence.
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A lower bound: directed column-convex permutominoes

To count directed column-convex permutominoes we have to restrict the operator $\varnothing$:

- Let $\varnothing'$ be the operator which performs the operations $\alpha$, $\beta$, and $\delta$ defined before (but not operation $\gamma$).
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Starting from the one cell permutomino, the operator $\vartheta'$ generates all column-convex permutominoes which do not contain any $WS$ reentrant point.

More precisely, the column-convex permutominoes which are *north-west directed* (briefly, directed).
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- More precisely, the column-convex permutominoes which are *north-west directed* (briefly, directed).
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Let $G(x, y, z)$ be the g.f. of directed column convex permutominoes, and $g(y) = G(y, y, y)$. Using the operator $\vartheta'$, we obtain

$$g(y) = 2y g(y) + y^2 g'(y) + y,$$

Proposition (Beaton, Disanto, Guttmann, R, 2010)

The number of directed column-convex permutominoes of size $n$ is $\frac{(n+1)!}{2}$. 
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Permutominides

(*Informal definition*) a *permutominide* is a set of cells:
- which can be edge-connected or vertex-connected
- having exactly one side for every abscissa and exactly one side for every ordinate.
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As a permutomino – a permutominide is defined by a pair of permutations.
An upper bound: column-convex permutominides

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Let us consider *column-convex permutominoes*

To a cc-permutominide $P$ associate a *side* permutation $\pi(P)$;

The permutation $\pi(P)$ does not uniquely determine $P$;
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An upper bound: column-convex permutominoes

**Proposition**

*Given a permutation* \( \sigma \) *of size* \( n + 1 \) *we have* \( 2^{n-2} \) *cc-permutominoes of size* \( n \) *having* \( \sigma \) *as side permutation.*

Let \( \Gamma \subseteq \{4, \ldots, n+1\} \): join \( \sigma(1) \) with \( \sigma(3) \) and in sequence all the points \( \sigma(i) \), with \( i \in \Gamma \), until reach the right side of the minimal bounding square; the permutominide is uniquely determined. Example: \( \Gamma = \{4, 7, 8\} \).
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![Diagram](image.png)
An upper bound: column-convex permutominides

Proposition

*The number of column convex permutominides of size $n$ is $2^{n-2}(n + 1)!$*
Tony Guttmann and Nick Beaton applying numerical analysis were led to conjecture that

\[ f_n \sim k (n + 1)! h^n, \]

where \( k = 0.3419111 \) and \( h = 1.385933 \).
In a convex permutomino of size $n$ reentrant points form a permutation of length $n - 1$;

We restrict the construction of $\vartheta$ to the class of convex permutominoes;

We obtain a \textit{factorial} generating tree.
Convex permutominoes

A generating tree for convex permutominoes

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The functional equation can be solved using the *kernel method*, and leads to an algebraic generating function.

The number of *convex permutominoes* of size $n$ is:

$$2 (n + 3) 4^{n-2} - \frac{n}{2} \binom{2n}{n} \quad n \geq 1,$$

(1)

the first few terms being 1, 4, 18, 84, 394, 1836, 8468, ...
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Another proof for the number of convex permutominoes

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- A different proof for the number of convex permutominoes has been obtained independently by Boldi, Lonati, Radicioni and Santini in 2007.
- Consists in three steps:
  - Encoding a convex permutomino of size $n$ in terms of a pair $(\nu, h)$: \( (\text{the transform of a permutation pair}) \)
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A bijective proof for the number of convex permutominoes

A bijective proof for the number of convex permutominoes has been given by Disanto, Duchi, Pinzani, R (2012).

Proposition

The number of convex permutominides of size $n$ is $2(n + 1)4^{n-2}$. 
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**Proposition**

*The number of convex permutominides of size $n$ is $2(n + 1)4^{n-2}$.*
A bijective proof for the number of convex permutominoes

Proposition

The number of convex permutominoes whose boundary crosses itself is equal to \( \frac{n}{2} \binom{2n}{n} - 4^{n-1} \).

By difference we obtain the formula for the number of convex permutominoes.
Proposition (Fanti, Frosini, Grazzini, Pinzani, R, 2007)

The number of parallelogram permutominoes of size \( n \) is equal to the \( n \)-th Catalan number,

\[
\frac{1}{n + 1} \binom{2n}{n}
\]
Proof.

A parallelogram permutomino of size $n$ is uniquely determined by its upper path,
which is a Dyck path of length $2n$;
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Parallelogram permutominoes

Kreweras involution

- The mapping of the upper path into the lower path of the parallelogram permutomino is an involution on Dyck paths:
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Directed-convex permutominoes

Proposition (Fanti, Frosini, Grazzini, Pinzani, R, 2007)

The number of directed convex permutominoes of size $n$ is equal to $\binom{2n-1}{n}$. 
Enumeration of permutominides

All the enumeration results have been obtained in bijective way, mapping the permutominide into a lattice path.

<table>
<thead>
<tr>
<th>permutominides</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>closed form</th>
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<tbody>
<tr>
<td>parallelogram</td>
<td>1</td>
<td>3</td>
<td>10</td>
<td>35</td>
<td>126</td>
<td>426</td>
<td>. .</td>
<td>$\binom{2n-1}{n}$</td>
</tr>
<tr>
<td>directed-convex</td>
<td>1</td>
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<td>16</td>
<td>64</td>
<td>256</td>
<td>1024</td>
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<td>$4^{n-1}$</td>
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<tr>
<td>convex</td>
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<td>32</td>
<td>160</td>
<td>768</td>
<td>3584</td>
<td>. .</td>
<td>$2(n+1)4^{n-2}$</td>
</tr>
<tr>
<td>row-convex</td>
<td>1</td>
<td>6</td>
<td>48</td>
<td>480</td>
<td>5768</td>
<td>80640</td>
<td>. .</td>
<td>$(n+1)!2^{n-2}$</td>
</tr>
</tbody>
</table>
Problem: characterizing permutations associated with permutominoes.

We start from the characterization of permutations associated with convex permutominoes.

\[
\pi_1 = (2, 1, 3, 4, 5, 7, 6) \\
\pi_2 = (2, 1, 3, 4, 5, 7, 6)
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\[
\pi_2 = (5, 1, 6, 7, 3, 2, 4)
\]
Permutations associated with convex permutominoes

Characterization of the following sets:

1. $\mathcal{P}_1(n) = \{\pi_1(P) : P \text{ is a convex permutomino of size } n\}$
2. $\mathcal{P}_2(n) = \{\pi_2(P) : P \text{ is a convex permutomino of size } n\}$
3. $\mathcal{A}(n) = \mathcal{P}_1(n) \cup \mathcal{P}_2(n)$

Example

A convex permutomino and the associated permutations
Permutations associated with convex permutominoes

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\[\pi_1 = (2 \ 5 \ 6 \ 1 \ 7 \ 3 \ 4)\]
Permutations associated with convex permutominoes

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Example

A convex permutomino and the associated permutations

\[
\pi_1 = (5 \ 6 \ 7 \ 2 \ 4 \ 1 \ 3)
\]

\[
\pi_2 = (5 \ 6 \ 7 \ 2 \ 4 \ 1 \ 3)
\]
Given a permutation $\pi$ we build its convex hull by joining the sequences of $lrmax$, $rlmax$, $rlmin$, and $lrmin$ of $\pi$.

Example

The convex hull of $(3, 8, 4, 6, 2, 9, 1, 2, 7)$
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The convex hull of $(3, 8, 4, 6, 2, 9, 1, 2, 7)$
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Square Permutations

Definition (Mansour, Severini, 2007)

A square permutation is a permutation without interior points. Also known as a convex permutation [Albert, Linton, Ruskuc, Vatter, Waton].
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**Example**

![Diagram showing simple fixed point](image-url)

*simple fixed point*
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A permutation is *decomposable* if it is the direct sum of (at least) two permutations;

A permutation is *m-decomposable* if its mirror image is decomposable.

**Lemma**

If a permutation $\pi$ is $m$-decomposable then there is no permutomino $P$ such that $\pi_1(P) = \pi$. 
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Let $[P]_{\sim} = \{ P' \text{ convex permutomino} : \pi_1(P) = \pi_1(P') \}$; then

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![Diagram of permutominoes](image_url)
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Proposition (Mansour, Severini, 2007)

The number of square permutations of length $n$ is

$$2(n + 2)4^{n-3} - 4(2n - 5)\binom{2(n - 3)}{n - 3}$$
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- Re-obtained by Duchi and Poulalhon (2008) by applying the ECO method;
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Some bijective problems

Relations between the number of convex permutominoes, square permutations, and convex polyominoes.
- $C_n$: number of convex permutominoes of size $n$,
- $Q_n$: number of square permutations of length $n$,
- $P_n$: number of convex polyominoes of semi-perimeter $n$;

1. $Q(n + 2) = C(n + 2) + \binom{2n}{n}$;
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Proposition (Albert, Linton, Ruskuc, Waton, 2011)

*The set of square permutations is the set of permutations avoiding 16 patterns of length 5:*

\[
(5 \ 2 \ 3 \ 4 \ 1) \ (5 \ 2 \ 3 \ 1 \ 4) \ (5 \ 1 \ 3 \ 4 \ 2) \ (5 \ 1 \ 3 \ 2 \ 4) \\
(4 \ 2 \ 3 \ 5 \ 1) \ (4 \ 2 \ 3 \ 1 \ 5) \ (4 \ 1 \ 3 \ 5 \ 2) \ (4 \ 1 \ 3 \ 2 \ 5) \\
(2 \ 5 \ 3 \ 4 \ 1) \ (2 \ 5 \ 3 \ 1 \ 4) \ (1 \ 5 \ 3 \ 4 \ 2) \ (1 \ 5 \ 3 \ 2 \ 4) \\
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Square Permutations and Pattern Avoidance

The 16 patterns describe the possibilities to have at least a point inside the convex hull of a permutation.

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More on square permutations

The records of a permutation

Open problem: enumerate permutations according to Irmin, rlmin, Irmax, and rlmax (the records of a permutation).


The number of Irmax is a standard statistic for permutations: the expected number of Irmax in a random permutation of large size $n$ is $\log n + O(n)$.

Enumerate permutations according to Irmax and rlmax is a rather simple exercise.
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ECO construction

- The ECO construction by Duchi, Poulalhon allows us to control the records of square permutations;
- Disanto, Duchi, Schaeffer, R (2011) modified such a construction in order to generate all permutations of length $n + 1$ starting from those of length $n$ and control the records of the permutations and the number of internal points.
- We have not been able to solve Wilf’s problem yet due to the complexity of the generating tree associated with the construction.
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Enumeration of permutations according to the number of internal points

Proposition (Disanto, Duchi, Schaeffer, R, 2011)

For all $i \geq 0$ the generating function of permutations with $i$ internal points is algebraic of degree 2 and it is a rational function in the Catalan generating function.

Conjecture

The number of permutations of length $n$ with $i$ internal points satisfies

$$S_n^{(i)} \sim_{n \to \infty} \frac{8 \cdot i!}{2^i (2i)!} \cdot n^{2i+1} \cdot 4^n.$$
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Square permutations are associated with convex permutominoes.

**Problem:** What kind of permutations are associated with column-convex permutominoes? And with permutominoes?

We provide a characterization of the set of permutations associated with permutominoes:

\[ \mathcal{A}(n) = \{ \pi \in S_n : \pi = \pi_1(P) \text{ or } \pi = \pi_2(P) \}, \text{ for a perm. } P \]
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Proposition (R, Socci 2012)

For every permutation $\pi \in S_n$ there is at least a column-convex permutomino $P$ such that $\pi = \pi_1(P)$ or $\pi = \pi_2(P)$, i.e. $A(n - 1) = S_n$.

Proof.

It consists in an algorithm for building a column-convex permutomino of size $n - 1$ from any given permutation $\pi \in S_n$. 
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Proof.

- Let $\pi \in S_n$, $A = (1, \pi(1))$, $B = (\pi^{-1}(n), n)$, $C = (n, \pi(n))$;
- We divide the points of $\pi$ in four regions, and individuate two new points $L$, $R$;
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Proof.

If \( y_L > y_R \) we start from \( B \) with an east step in clockwise order connecting all the points in each of the four regions; otherwise we start from \( B \) with a west step.
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Permutations associated with permutominoes

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- The algorithm returns a column-convex permutomino.
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Further works and open problems

The number of column-convex permutominoes

- **Solve the functional equation:**

  \[
  F(x, y, z) = y + \frac{yz}{z-y} F(x, z, z) - \frac{y^2}{z-y} F(x, y, y) + \frac{xy}{x-y} F(x, x, z) - \frac{y^2}{x-y} F(y, y, z).
  \]

  and determine the number of column-convex permutominoes of size \( n \).

- **Prove that the number of column-convex permutominoes of size \( n \) behaves like:**

  \[
  f_n \sim k (n + 1)! h^n,
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  where \( k = 0.3419111 \) and \( h = 1.385933 \).
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Permutominoes enumeration

- Give an asymptotic estimation for the number of permutominoes.
- Enumerate permutominoes according to the number of cells (area).
- Enumeration of non convex permutominoes according to the semi-perimeter.
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We have considered 3-dimensional permutominoes: 3-dimensional polygons having exactly two sides of the boundary for each plane parallel to a coordinate plane. As expected they are uniquely defined by three 3-dimensional permutations. Work in progress with Duchi, Schaeffer: definition, combinatorial properties and enumeration of 3-dimensional permutominoes.
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Polyominoes determined by permutations

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A talk by D. Eppstein

The Topology of Bendless Orthogonal Three-Dimensional Graph Drawing

David Eppstein
Computer Science Dept.
Univ. of California, Irvine
Polyominoes determined by permutations

Further work