Polyominoes determined by permutations

Simone Rinaldi¹

¹University of Siena, Siena, Italy

GASCom, 2012



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- Permutominoes Enumeration
 - Column-convex permutominoes
 - A generating tree for column-convex permutominoes
 - Enumeration of column-convex permutominoes
 - Numerical analysis
 - Convex permutominoes
 - Enumeration through the ECO method
 - Bijective enumeration of some classes of convex permutominoes

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 - Permutations associated with convex permutominoes
 - Square permutations
 - Permutations and permutominoes

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- Permutations associated with convex permutominoes
- Square permutations
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Permutomino: working definition

Definition

A permutomino of size *n* is a polyomino (with no holes) having *n* rows and *n* columns, such that for each abscissa (ordinate) between 1 and n + 1 there is exactly one vertical (horizontal) bond in the boundary of *P* with that coordinate.



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Permutomino: equivalent definition

A permutomino *P* of size *n* is uniquely defined by a pair of permutations of size n + 1, denoted by $\pi_1(P)$ and $\pi_2(P)$, called permutations associated with *P*.



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Permutations associated with permutominoes

A pair of permutations (π_1, π_2) such that $\pi_1(i) \neq \pi_2(i)$ may not define a permutomino.

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The boundary may be non connected;

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Permutominoes: the origins

The singular locus of a Schubert variety

- Introduced in 2003 by Kassel, Lascoux, and Reutenauer, with the aim of characterizing the singular loci of the Schubert varieties X_μ in the flag variety for GL_n(C).
- The name *Permutomino* (or *'Permutaomino'*) was coined by Incitti (2006), while studying the problem of determining the \tilde{R} -polynomials associated with a pair (μ, ν) of permutations.

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Permutominoes: the origins

Graph of a permutation

The graph of a permutation $\mu \in S_n$ is the set of points $(i, \mu(i))$, with $i \in [n] = \{1, ..., n\}$.



Figure: Graph of $\mu = (3, 1, 5, 4, 7, 2, 9, 8, 6)$

Permutominoes: the origins

Bruhat order

For each $h, k \in [n]$ we set

$$\mu[h, k] = |\{i \in [n] : i \le h, \mu(i) \ge k\}|$$

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Permutominoes: the origins

Bruhat order

Given $\mu, \nu \in S_n$, and $h, k \in [n]$ we set:

$$(\mu, \nu)[h, k] =
u(h, k) - \mu(h, k)$$



Permutominoes: the origins

Proposition (Björner, Brenti, 2005)

Given $\mu, \nu \in S_n$, $\mu \leq \nu$ in the Bruhat order iff $(\mu, \nu)[h, k] \geq 0$, for all $(h, k) \in \mathbb{N}^2$.

Permutominoes: the origins

Planar representation of a pair of permutations

The *planar representation* of (μ, ν) is obtained as follows:

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Permutominoes: the origins

Planar representation of a pair of permutations

The *planar representation* of (μ, ν) is obtained as follows:

- we use the graphs of μ , ν ,
- if $\mu(i) = \nu(i)$ we have a double point,
- we connect the other points using horizontal and vertical lines.

Permutominoes: the origins



Figure: Planar representation of (μ, ν) , with $\mu = (3, 1, 5, 4, 7, 2, 9, 8, 6), \nu = (7, 8, 2, 4, 9, 6, 3, 1, 5)$

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Permutominoes: the origins

Planar representation of a pair of permutations

Bijective correspondence between the disjoint cycles of $\mu^{-1}\nu$ and the closed paths in the planar representation of (μ, ν) .



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Permutominoes: the origins

Diagram of a pair of permutations

Given $\mu, \nu \in S_n$, the *multiplicity mapping* of (μ, ν) :

 $(\mu, \nu)[h, k] = \nu(h, k) - \mu(h, k), \text{ with } (h, k) \in \mathbb{R}^2$



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Permutominoes: the origins

Permutominoes

A pair (μ, ν) is a *permutomino* if

- $(\mu, \nu)[h, k] \in \{0, 1\}$, for all $(h, k) \in \mathbb{R}^2$;
- (μ, ν) has no double points;
- the support of (μ, ν) is connected and with no holes.



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Permutominoes: the origins

Generation of orthogonal polygons

- Permutominoes have been considered independently by Tomás, Bajuelos (2004);
- Quadratic-time algorithm for the random generation of orthogonal polygons with a fixed number of vertices;
- Orthogonal polygons: simple polygons without holes where the edges meet at right angles.

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Permutominoes: the origins

• Generic orthogonal polygons with *n* vertices may be obtained from *grid n-ogons*: *n*-vertex orthogonal polygons having exactly one edge for every line of the grid.

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Permutominoes: the origins

Generation of orthogonal polygons

• Tomás, Bajuelos (2004) provide a quadratic-time algorithm for the random generation of grid *n*-ogons by means of the *INFLATE-PASTE* algorithm.



Permutominoes: the origins

Generation of orthogonal polygons

• Tomás, Bajuelos (2004) provide a quadratic-time algorithm for the random generation of grid *n*-ogons by means of the *INFLATE-PASTE* algorithm.

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Contents of the talk

- Enumeration of (various classes of) permutominoes according to the size;
- Characterization and enumeration of pairs of permutations defining (various classes of) permutominoes.



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Permutominoes enumeration

We consider the enumeration of some classes of permutominoes:

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- column-convex permutominoes,
- convex permutominoes,
- directed permutominoes

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Column-convex permutominoes

- We present an algorithm for the recursive generation of column-convex permutominoes such each object of size n + 1 is obtained in a unique way from an object of size n.
- We follow the ECO method, Enumeration of Combinatorial Objects developed by Pinzani and his team starting from '90.
- ECO method has been applied to several problems of enumeration and generation of combinatorial objects.
- It is strictly related to the concept of *generating tree* introduced by West (1994).

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Column-convex permutominoes

Generation of column-convex permutominoes

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- Permutominoes Enumeration
 - Column-convex permutominoes

Reentrant points

- Encode the boundary of the permutomino using north, east, south and west unit steps.
- Any occurrence of a sequence NE, ES, SW, or WN in the word encoding P defines a *salient point* of P; any occurrence of a sequence EN, SE, WS, or NW defines a *reentrant point* of P.



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Column-convex permutominoes

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- Encode the boundary of the permutomino using north, east, south and west unit steps.
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Column-convex permutominoes

Reentrant points

The difference between the number of salient and reentrant points in any closed path is equal to 4. In a column-convex permutomino of size n we have exactly one reentrant point for each abscissa between 1 and n - 1.



- Permutominoes Enumeration
 - Column-convex permutominoes

The generating operator

- Let C_n be the set of column-convex permutominoes of dimension *n*.
- We present a construction such that each element of C_{n+1} is produced from exactly one element of C_n.
- The idea is to let permutominoes grow from size n to n + 1 by adding a new reentrant point on the right.
- Our construction consists of four operations, $\alpha, \beta, \gamma, \delta$:
 - a adds an EN-reentrant point
 d adds a SE-reentrant point;
 d adds a WS-reentrant point;
 adds a WS-reentrant point;
 adds a NW-reentrant point;

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 - α adds an *EN*-reentrant point;
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 α adds an EV-reentrant point; β adds a SE-reentrant point; γ adds a WS-reentrant point; δ adds a WW-reentrant point;

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Permutominoes Enumeration

Column-convex permutominoes

Operation α



 $(h, d, w) \rightsquigarrow (h - i + 1, d + i, w)$ $1 \le i \le h + 1$

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Permutominoes Enumeration

Column-convex permutominoes

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Permutominoes Enumeration

Column-convex permutominoes

Operation β



 $(h, d, w) \rightsquigarrow (d + h - i + 1, i, w) \qquad 1 \le i \le d$

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Permutominoes Enumeration

Column-convex permutominoes

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Permutominoes Enumeration

Column-convex permutominoes

Operation γ



 $(h, d, w) \rightsquigarrow (h, i, d + w - i + 1)$ $1 \le i \le d$

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Permutominoes Enumeration

Column-convex permutominoes

Operation γ



 $(h, d, w) \rightsquigarrow (h, i, d + w - i + 1)$ $1 \le i \le d$

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Permutominoes Enumeration

Column-convex permutominoes

Operation δ



 $(h, d, w) \rightsquigarrow (h, d+i, w-i+1)$ $1 \le i \le w+1$

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Permutominoes Enumeration

Column-convex permutominoes

Operation δ



 $(h, d, w) \rightsquigarrow (h, d+i, w-i+1)$ $1 \le i \le w+1$

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Column-convex permutominoes

The generating operator

For a column-convex permutomino *P* we denote $\vartheta(P)$ the set of permutominoes produced by *P* through the application of α , β , γ , and δ .

Theorem (Beaton, Disanto, Guttmann, R, 2010)

Any column-convex permutomino of size $n \ge 2$ is uniquely obtained through the application of the operator ϑ to a column-convex permutomino of size n - 1.

Column-convex permutominoes

The generating tree

The recursive construction of ϑ can be formally described using generating trees (or succession rules). The application of ϑ to a column-convex permutomino with a generic label (h, d, w), produces 2d + h + w + 2 column-convex permutominoes according to the following rule:

$$\begin{array}{cccc} (h,d,w) & \stackrel{\alpha}{\longrightarrow} & (h-i+1,d+i,w) & 1 \leq i \leq h+1 \\ & \stackrel{\beta}{\longrightarrow} & (d+h-i+1,i,w) & 1 \leq i \leq d \\ & \stackrel{\gamma}{\longrightarrow} & (h,i,d+w-i+1) & 1 \leq i \leq d \\ & \stackrel{\delta}{\longrightarrow} & (h,d+i,w-i+1) & 1 \leq i \leq w+1. \end{array}$$

with root (0, 1, 0), which is the label of the one cell permutomino.

Permutominoes Enumeration

Column-convex permutominoes

The functional equation

The construction of operator ϑ can be translated onto generating functions:

$$F(x, y, z) = \sum_{P \in C} x^{h(P)} y^{d(P)} z^{w(P)}$$

= $y + xy + yz + 2y^2 + 3x^2y + 2xyz + 3z^2y + 4xy^2 + 4zy^2 + 6y^3 + \dots$

$$\begin{split} F(x, y, z) &= y + \sum_{P \in \mathcal{C}} x^h y^{d+1} z^w + \ldots + x^0 y^{d+h+1} z^w + \sum_{P \in \mathcal{C}} x^{d+h} y^1 z^w + \ldots + x^{h+1} y^d z^w \\ &+ \sum_{P \in \mathcal{C}} x^h y z^{d+w} + \ldots + x^h y^d z^{w+1} + \sum_{P \in \mathcal{C}} x^h y^{d+1} z^w + \ldots + x^h y^{d+w+1} z^0, \end{split}$$

Permutominoes Enumeration

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Permutominoes Enumeration

Column-convex permutominoes

The functional equation

We obtain the following functional equation

$$F(x, y, z) = y + \frac{yz}{z - y}F(x, z, z) - \frac{y^2}{z - y}F(x, y, y) + \frac{xy}{x - y}F(x, x, z) - \frac{y^2}{x - y}F(y, y, z).$$

 From this equation we are able to compute the first 200 terms of sequence f_n beginning with:

 $1, 4, 22, 152, 1262, 12232, 135544, 1690080, 23417928, \ldots$

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Permutominoes Enumeration

Column-convex permutominoes

A lower bound: directed column-convex permutominoes

To count directed column-convex permutominoes we have to restrict the operator ϑ :

 Let θ' be the operator which performs the operations α, β, and δ defined before (but not operation γ).

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- Permutominoes Enumeration

Column-convex permutominoes

A lower bound: directed column-convex permutominoes

- Starting from the one cell permutomino, the operator θ' generates all column-convex permutominoes which do not contain any WS reentrant point.
- More precisely, the column-convex permutominoes which are north-west directed (briefly, directed).



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Permutominoes Enumeration

Column-convex permutominoes

A lower bound: directed column-convex permutominoes

Let G(x, y, z) be the g.f. of directed column convex permutominoes, and g(y) = G(y, y, y). Using the operator ϑ' , we obtain

$$g(y) = 2y g(y) + y^2 g'(y) + y$$
,

Proposition (Beaton, Disanto, Guttmann, R, 2010)

The number of directed column-convex permutominoes of size n is $\frac{(n+1)!}{2}$.

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Column-convex permutominoes

An upper bound: column-convex permutominides

Permutominides

(Informal definition) a permutominide is a set of cells:

- which can be edge-connected or vertex-connected
- having exactly one side for every abscissa and exactly one side for every ordinate.



Column-convex permutominoes

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Column-convex permutominoes

An upper bound: column-convex permutominides

Permutominides

As a permutomino – a permutominide is defined by a pair of permutations.


Column-convex permutominoes

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Permutominides

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- To a cc-permutominide *P* associate a *side* permutation π(*P*);
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Column-convex permutominoes

An upper bound: column-convex permutominides

Proposition

Given a permutation σ of size n + 1 we have 2^{n-2} cc-permutominides of size n having σ as side permutation.

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Column-convex permutominoes

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Let $\Gamma \subseteq \{4, \ldots, n+1\}$: join $\sigma(1)$ with $\sigma(3)$ and in sequence all the points $\sigma(i)$, with $i \in \Gamma$, until reach the right side of the minimal bounding square; the permutominide is uniquely determined. Example: $\Gamma = \{4, 7, 8\}$.



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Column-convex permutominoes

An upper bound: column-convex permutominides

Proposition

The number of column convex permutominides of size n is $2^{n-2}(n+1)!$

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Permutominoes Enumeration

Column-convex permutominoes

Numerical approach

Tony Guttmann and Nick Beaton applying numerical analysis were led to conjecture that

 $f_n \sim k \, (n+1)! \, h^n \, ,$

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where k = 0.3419111 and h = 1.385933.

- Permutominoes Enumeration
 - Convex permutominoes

Convex permutominoes

A generating tree for convex permutominoes

 In a convex permutomino of size *n* reentrant points form a permutation of length *n* − 1;

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- We restrict the construction of *θ* to the class of convex permutominoes;
- We obtain a factorial generating tree.

- Permutominoes Enumeration
 - Convex permutominoes

Convex permutominoes

A generating tree for convex permutominoes

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Permutominoes Enumeration

Convex permutominoes

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- Permutominoes Enumeration

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Permutominoes Enumeration

└─ Convex permutominoes

Enumeration of convex permutominoes

• The functional equation can be solved using the *kernel method*, and leads to an algebraic generating function.

The number of convex permutominoes of size n is :

$2(n+3)4^{n-2} - \frac{n}{2}\binom{2n}{n} \qquad n \ge 1,$ (1)

the first few terms being 1, 4, 18, 84, 394, 1836, 8468, ...

Convex permutominoes

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- Permutominoes Enumeration
 - └─ Convex permutominoes

Exhaustive generation of convex permutominoes

Exhaustive generation of convex permutominoes

 Using the ECO construction Grazzini, Pergola and Poneti (2008) determined an algorithm for the exhaustive generation of convex permutominoes of given size;
 The algorithm works in Constant Amortized Time.

- Permutominoes Enumeration
 - Convex permutominoes

Exhaustive generation of convex permutominoes

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The algorithm works in Constant Amortized Time.

- Permutominoes Enumeration
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Permutominoes Enumeration

└─ Convex permutominoes

Another proof for the number of convex permutominoes

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Consists in three steps:

Permutominoes Enumeration

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Permutominoes Enumeration

Convex permutominoes

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- Consists in three steps:
 - encoding a convex permutomino of size n in terms of a pair of permutations (ν, h): (the transform of a permutation pair);
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Permutominoes Enumeration

└─ Convex permutominoes

A bijective proof for the number of convex permutominoes

A bijective proof for the number of convex permutominoes has been given by Disanto, Duchi, Pinzani, R (2012).

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Proposition

The number of convex permutominides of size *n* is $2(n+1)4^{n-2}$.
- Permutominoes Enumeration

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- Permutominoes Enumeration

Convex permutominoes

A bijective proof for the number of convex permutominoes

Proposition

The number of convex permutominides whose boundary crosses itself is equal to $\frac{n}{2} \binom{2n}{n} - 4^{n-1}$.

By difference we obtain the formula for the number of convex permutominoes.



Permutominoes Enumeration

Convex permutominoes

Parallelogram permutominoes

Proposition (Fanti, Frosini, Grazzini, Pinzani, R, 2007)

The number of parallelogram permutominoes of size n is equal to the n-th Catalan number,

$$\frac{1}{n+1}\binom{2n}{n}$$

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Permutominoes Enumeration

Convex permutominoes

Parallelogram permutominoes

Proof.

 A parallelogram permutomino of size n is uniquely determined by its upper path,

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which is a Dyck path of length 2n;

Permutominoes Enumeration

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Permutominoes Enumeration

Convex permutominoes

Parallelogram permutominoes

Kreweras involution

 The mapping of the upper path into the lower path of the parallelogram permutomino is an involution on Dyck paths:

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 the so-called Kreweras involution on Dyck paths [Kreweras, 1970];

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Permutominoes Enumeration

Convex permutominoes

Directed-convex permutominoes

Proposition (Fanti, Frosini, Grazzini, Pinzani, R, 2007)

The number of directed convex permutominoes of size *n* is equal to $\binom{2n-1}{n}$.

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Permutominoes Enumeration

Convex permutominoes

Enumeration of permutominides

All the enumeration results have been obtained in bijective way, mapping the permutominide into a lattice path.

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permutominides	1	2	3	4	5	6	7	closed form
parallelogram	1	3	10	35	126	426		$\binom{2n-1}{n}$
directed-convex	1	4	16	64	256	1024		4 ^{<i>n</i>-1}
convex	1	6	32	160	768	3584		$2(n+1)4^{n-2}$
row-convex	1	6	48	480	5768	80640		$(n+1)! 2^{n-2}$

Permutations associated with permutominoes

Permutations associated with permutominoes

- Problem: characterizing permutations associated with permutominoes.
- We start from the characterization of permutations associated with convex permutominoes.





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- Permutations associated with permutominoes
 - Permutations associated with convex permutominoes

Characterization of the following sets:

- $\mathcal{P}_1(n) = \{\pi_1(P) : P \text{ is a convex permutomino of size } n\}$
- \$\mathcal{P}_2(n) = {\$\pi_2(P) : P\$ is a convex permutomino of size \$n\$}\$
 \$\mathcal{A}(n) = \$\mathcal{P}_1(n) \cup \$\mathcal{P}_2(n)\$

Example

A convex permutomino and the associated permutations

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- Permutations associated with permutominoes
 - -Square permutations

Square permutations

Given a permutation π we build its convex hull by joining the sequences of *Irmax*, *rImax*, *rImin*, and *Irmin* of π .



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Square Permutations

Definition (Mansour, Severini, 2007)

A square permutation is a permutation without interior points. Also known as a convex permutation [Albert, Linton, Ruskuc, Vatter, Waton].



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Permutations associated with permutominoes

Square permutations

Fixed Points in a Square Permutation

In a square permutation we have two types of fixed points
Non extremal fixed points that belong to two faces are called double points.

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Decomposable square permutations

A permutation is *decomposable* if it is the direct sum of (at least) two permutations;

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A permutation is *m-decomposable* if its mirror image is decomposable.

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If a permutation π is m-decomposable then there is no permutomino P such that $\pi_1(P) = \pi$.
Permutations associated with permutominoes

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Proposition (Fanti, Frosini, Grazzini, Pinzani, R, 2007)

Let $[P]_{\sim} = \{P' \text{ convex permutomino } : \pi_1(P) = \pi_1(P')\}$; then

 $|[P]_{\sim}| = 2^{d(P)},$

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On square permutations

Proposition (Mansour, Severini, 2007)

The number of square permutations of length n is

$$2(n+2)4^{n-3}-4(2n-5)\binom{2(n-3)}{n-3}$$

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On square permutations

Some bijective problems

Relations between the number of convex permutominoes, square permutations, and convex polyominoes.

- *C_n*: number of convex permutominoes of size *n*,
- Q_n: number of square permutations of length n,
- *P_n*: number of convex polyominoes of semi-perimeter *n*;

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$$Q(n+2) = C(n+2) + \binom{2n}{n};$$

2 $P(n+2) = Q(n+1) + 4^{n-2};$

Give a bijective proof for the number of square permutations (all the known proofs use analytical methods)

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Square Permutations and Pattern Avoidance

Proposition (Albert, Linton, Ruskuc, Waton, 2011)

The set of square permutations is the set of permutations avoiding 16 patterns of length 5:

Permutations associated with permutominoes

Square permutations

Square Permutations and Pattern Avoidance

The 16 patterns describe the possibilities to have at least a point inside the convex hull of a permutation.



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Permutations associated with permutominoes

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More on square permutations

- **Open problem**: enumerate permutations according to Irmin, rlmin, Irmax, and rlmax (the *records* of a permutation).
- Proposed by H. Wilf at the *"Conference on Probabilistic Methods and Enumeration"* CRM, Barcelona, 2007.
- The number of Irmax is a standard statistic for permutations: the expected number of Irmax in a random permutation of large size n is log n + O(n).
- Enumerate permutations according to Irmax and rImax is a rather simple exercise.

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ECO construction

- The ECO construction by Duchi, Poulalhon allows us to control the records of square permutations;
- Disanto, Duchi, Schaeffer, R (2011) modified such a construction in order to generate all permutations of length n+1 starting from those of length n and control the records of the permutations and the number of internal points.
- We have not been able to solve Wilf's problem yet due to the complexity of the generating tree associated with the construction.

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Enumeration of permutations according to the number of internal points

Proposition (Disanto, Duchi, Schaeffer, R, 2011)

For all $i \ge 0$ the generating function of permutations with *i* internal points is algebraic of degree 2 and it is a rational function in the Catalan generating function.

Conjecture

The number of permutations of length n with i internal points satisfies

$$\left|S_n^{(i)}\right| \sim_{n \to \infty} \frac{8 \cdot i!}{2^i (2i)!} \cdot n^{2i+1} \cdot 4^n.$$

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Permutations associated with permutominoes

Permutations associated with permutominoes

- Square permutations are associated with convex permutominoes.
- Problem: What kind of permutations are associated with column-convex permutominoes? And with permutominoes?
- We provide a characterization of the set of permutations associated with permutominoes:

 $\mathcal{A}(n)=\{\pi\in S_n:\pi=\pi_1(P) ext{ or }\pi=\pi_2(P), ext{ for a perm. }P\}$

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Permutations associated with permutominoes

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Proposition (R, Socci 2012)

For every permutation $\pi \in S_n$ there is at least a column-convex permutomino P such that $\pi = \pi_1(P)$ or $\pi = \pi_2(P)$, i.e. $\mathcal{A}(n-1) = S_n$.

Proof.

It consists in an algorithm for building a column-convex permutomino of size n - 1 from any given permutation $\pi \in S_n$.

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 - Permutations and permutominoes

Permutations associated with permutominoes

Proof.

- Let $\pi \in S_n$, $A = (1, \pi(1))$, $B = (\pi^{-1}(n), n) C = (n, \pi(n))$;
- We divide the points of π in four regions, and individuate two new points L, R;

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Permutations associated with permutominoes

Proof.

 If y_L > y_R we start from B with an east step in clockwise order connecting all the points in each of the four regions; otherwise we start from B with a west step.

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Proof.

• The algorithm returns a column-convex permutomino.

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• If π is square it is a convex permutomino.

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• If π is square it is a convex permutomino.

Further works and open problems

The number of column-convex permutominoes

• Solve the functional equation:

$$F(x, y, z) = y + \frac{yz}{z - y} F(x, z, z) - \frac{y^2}{z - y} F(x, y, y) + \frac{xy}{x - y} F(x, x, z) - \frac{y^2}{x - y} F(y, y, z).$$

and determine the number of column-convex permutominoes of size *n*.

• Prove that the number of column-convex permutominoes of size *n* behaves like:

$$f_n \sim k(n+1)! h^n,$$

where k = 0.3419111 and h = 1.385933.

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- Give an asymptotic estimation for the number of permutominoes.
- Enumerate permutominoes according to the number of cells (area).
- Enumeration of non convex permutominoes according to the semi-perimeter.

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- We have considered 3-dimensional permutominoes:
 3-dimensional polygons having exactly two sides of the boundary for each plane parallel to a coordinate plane.
- As expected they are uniquely defined by three 3-dimensional permutations.
- Work in progress with Duchi, Schaeffer: definition, combinatorial properties and enumeration of 3-dimensional permutominoes.

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Further work

Extension of permutominoes

A talk by D. Eppstein

The Topology of Bendless Orthogonal Three-Dimensional Graph Drawing



David Eppstein Computer Science Dept. Univ. of California, Irvine

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-Further work



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-Further work



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