

# The identity transform of a permutation

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# Hall's Theorem

## Theorem (Hall's Theorem)

Let  $G$  be an abelian additive group of order  $n$   $G = \{a_1, \dots, a_n\}$  and let  $c_1, \dots, c_n$  be elements of  $G$ , not necessarily distinct, then there is a permutation  $\pi$  of  $\{a_1, \dots, a_n\}$  such that  $a_i + \pi(a_i) = c_i$  for  $i = 1, \dots, n$ , where  $c_i$  are reordered properly, if and only if

$$\sum_{i=0}^n c_i = 0.$$

# Hall's Theorem

## Example

We can consider  $G = \mathbb{Z}_5$  and let  $C = \{0, 2, 2, 3, 3\}$  be a multiset of  $G$  of length 5. It holds  $\sum_{i=0}^4 c_i = 2 \cdot 5 \equiv_5 0$ , then, according to Hall's Theorem, follows:

$$\begin{array}{c|ccccc} id & 0 & 1 & 2 & 3 & 4 \\ \pi_1 & 3 & 2 & 0 & 4 & 1 \\ \hline C_1 & 3 & 3 & 2 & 2 & 0 \end{array}$$

$$\begin{array}{c|ccccc} id & 0 & 1 & 2 & 3 & 4 \\ \pi_2 & 2 & 1 & 3 & 0 & 4 \\ \hline C_2 & 2 & 2 & 0 & 3 & 3 \end{array}$$

where  $C_1$  and  $C_2$  are rearrangements of  $C$ .

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$id$	0	1	2	3	4
$\pi_1$	3	2	0	4	1
$C_1$	3	3	2	2	0

$id$	0	1	2	3	4
$\pi_2$	2	1	3	0	4
$C_2$	2	2	0	3	3

where  $C_1$  and  $C_2$  are rearrangements of  $C$ .

## Remarks

- the rearrangement is not unique;

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$$\begin{array}{c|ccccc} id & 0 & 1 & 2 & 3 & 4 \\ \pi_1 & 3 & 2 & 0 & 4 & 1 \\ \hline C_1 & 3 & 3 & 2 & 2 & 0 \end{array} \qquad \begin{array}{c|ccccc} id & 0 & 1 & 2 & 3 & 4 \\ \pi_2 & 2 & 1 & 3 & 0 & 4 \\ \hline C_2 & 2 & 2 & 0 & 3 & 3 \end{array}$$

where  $C_1$  and  $C_2$  are rearrangements of  $C$ .

## Remarks

- the rearrangement is not unique;
- the proof given by Hall is a **NON CONSTRUCTIVE** proof, since it uses the *reductio ad absurdum*.

## Goal

Find an alternative **CONSTRUCTIVE** proof of Hall's Theorem.

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Find an alternative **CONSTRUCTIVE** proof of Hall's Theorem.

## What we do

- We define a new set  $C_n$ .
- We find some combinatorial properties of  $C_n$ .
- We give an idea for a new approach to Hall's Theorem.

# The identity transform of a permutation

## Definition

Let  $\pi$  be a permutation, the *identity transform* of  $\pi$  is the vector:

$$C(\pi) = (0 + \pi(0), 1 + \pi(1), \dots, (n - 1) + \pi(n - 1)).$$

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## Example

$id$		0	1	2	3	4
$\pi$		3	4	1	2	0
$C(\pi)$		3	0	3	0	4

# The set $C_n$

## Definition

We define  $C_n$  as:

$$C_n = \{C(\pi) : \pi \in S_n\},$$

where  $S_n$  denotes the set of permutations of length  $n$ .

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## Remarks

- There is a bijective correspondence between the elements in  $C_n$  and the permutations of length  $n$ .
- If a vector  $C = (c_0, \dots, c_{n-1}) \in C_n$  then  $\sum_{i=0}^{n-1} c_i \equiv_n 0$ , but the converse does not hold.

For example, with  $n = 5$ , one can check that the vector  $(1, 2, 3, 0, 4)$  sums up to 0 modulo 5, but it is not an element of  $C_5$ .

# Combinatorial properties of $C_n$

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- Clousure properties of  $C_n$ 
  - cyclic shift;
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- Characterizations of  $C_n$ 
  - fixed points;
  - switch;
  - slide.

# Cyclic shift

## Definition

Let  $C = (c_0, \dots, c_{n-1})$ , its  $i$ -th cyclic shift  $S^i(C)$  is the vector  
 $S^i(C) = (c_{n-i}, \dots, c_0, \dots, c_{n-i-1})$ .

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## Example

Let  $C = (1, 1, 1, 0, 2)$  be a vector,  $S^3(C) = (1, 0, 2, 1, 1)$ .

# Cyclic shift

## Theorem

*If  $C \in C_n$  then  $S^i(C) \in C_n$ , for all  $i \leq n - 1$ .*

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## Example

Let  $C = (1, 1, 1, 0, 2) \in C_5$ :

$id$		0	1	2	3	4
$\pi$		1	0	4	2	3
$C(\pi)$		1	1	1	0	2

$id$		0	1	2	3	4
$\pi'$		2	0	4	3	1
$C(\pi')$		2	1	1	1	0

Then the vector  $S^1(C) = (2, 1, 1, 1, 0) \in C_5$ .

# Translation

## Definition

Let  $C = (c_0, \dots, c_{n-1})$ , we call  $C^{+i} = C + i = (c_0 + i, \dots, c_{n-1} + i)$  the *i-th translation* of  $C$ .

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## Example

Let  $C = (1, 1, 1, 0, 2)$  be a vector,  $C^{+3} = (4, 4, 4, 3, 0)$ .



# Translation

## Theorem

*If  $C \in C_n$  then  $C^{+i} \in C_n$ , for each  $i = 1, \dots, n - 1$ .*

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## Example

Let  $C = (1, 1, 1, 0, 2) \in C_5$ :

$id$		0	1	2	3	4
$\pi$		1	0	4	2	3
$C(\pi)$		1	1	1	0	2

$id$		0	1	2	3	4
$\pi + 3$		4	3	2	0	1
$C^{+3}$		4	4	4	3	0

Then the vector  $C^{+3} = (4, 4, 4, 3, 0) \in C_5$ .

# Switch

## Definition

Let  $C = (c_0, \dots, c_{n-1})$ , we say that

$\psi_i(C) = (c_0, \dots, c_{i+1} - 1, c_i + 1, \dots, c_{n-1})$  is obtained from  $C$  by a *switch* of index  $i$ .

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## Example

Let  $C = (1, 1, \mathbf{1}, \mathbf{0}, 2)$  be a vector,  $\psi_2(C) = (1, 1, \mathbf{4}, \mathbf{2}, 2)$ .

# Switch

## Theorem

*If  $C \in C_n$  then  $\psi_i(C) \in C_n$ .*

# Switch

## Theorem

If  $C \in C_n$  then  $\psi_i(C) \in C_n$ .

## Example

Let  $C = (1, 1, 1, 0, 2) \in C_5$

$id$		0	1	2	3	4
$\pi$		1	0	<b>4</b>	<b>2</b>	3
$C(\pi)$		1	1	<b>1</b>	<b>0</b>	2

$id$		0	1	2	3	4
$\pi'$		1	0	<b>2</b>	<b>4</b>	3
$C(\pi')$		1	1	<b>4</b>	<b>2</b>	2

Hence  $\psi_2(C) = (1, 1, 4, 2, 2) \in C_5$ .

# Anti-exceedances and Eulerian numbers

## Definition

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## Example

The anti-exceedances of  $C = (1, 1, 1, 0, 2) \in C_5$  are  $i = 2, 3, 4$ :

$id$		0	1	2	3	4
$\pi$		1	0	4	2	3
$C(\pi)$		1	1	<b>1</b>	<b>0</b>	<b>2</b>



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$id$		0	1	2	3	4
$\pi$		1	0	4	2	3
$C(\pi)$		1	1	<b>1</b>	<b>0</b>	<b>2</b>

## Remark

Observe that if an index  $i$  is an anti-exceedance then  $i + \pi(i) \geq n$ .

# Anti-exceedances and Eulerian numbers

## Theorem

Let  $\pi$  be a permutation and let  $k$  be the number of the anti-exceedances of  $C(\pi) = (c_0, \dots, c_{n-1})$  and let  $\sum_{i=0}^{n-1} c_i = l \cdot n$  then  $k + l = n - 1$ .

# Anti-exceedances and Eulerian numbers

## Theorem

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## Example

In the vector  $(7, 7, 4, 2, 5, 8, 6, 3, 3)$  the number of anti-exceedances plus  $l = 5$  (since  $\sum_{i=0}^{n-1} c_i = 9 \cdot 5$ ) gives  $n - 1 = 8$ .

$id$	0	1	2	3	4	5	6	7	8
$\pi$	7	6	2	8	1	3	0	5	4
$C(\pi)$	7	7	4	2	5	8	6	3	3

# Anti-exceedances and Eulerian numbers

## Theorem

*The number  $C_{n,k}$  of vectors of  $C_n$  with  $k$  anti-exceedances is given by the Eulerian number  $E_{n,k}$ .*

We find a correspondence between the set of vectors of  $C_n$  with  $k$  anti-exceedances and the set of permutations of  $S_n$  with  $k$  exceedances, which is enumerated by Eulerian numbers  $E_{n,k}$ .

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We find a correspondence between the set of vectors of  $C_n$  with  $k$  anti-exceedances and the set of permutations of  $S_n$  with  $k$  exceedances, which is enumerated by Eulerian numbers  $E_{n,k}$ .

## Example

Let  $C(\pi) = (1, 1, 1, 0, 2) \in C_5$  with three anti-exceedances, then we build the corresponding permutation  $\pi'$  (the mirror image of  $\pi$ ) with three exceedances by the bijection:

$$\begin{array}{c|ccccc} id & 0 & 1 & 2 & 3 & 4 \\ \pi & 1 & 0 & 4 & 2 & 3 \\ \hline C(\pi) & 1 & 1 & \mathbf{1} & \mathbf{0} & \mathbf{2} \end{array} \leftrightarrow \begin{array}{c|ccccc} id & 0 & 1 & 2 & 3 & 4 \\ \pi' & \mathbf{3} & \mathbf{2} & \mathbf{4} & 0 & 1 \end{array}$$

## Combinatorial characterizations of $C_n$

An index  $s$  is a fixed point of  $C$  if  $s = c_s$ .

### Theorem

*Let  $C$  be a vector and let  $S^i(C)$  be the its  $i$ -th cyclic shift. Then,  $C \in C_n$  if and only if  $S^i(C)$  has a unique fixed point for all  $i \leq n$ .*

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## Theorem

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## Example

$C$	5	3	<b>2</b>	4	1	3
$S^1(C)$	3	5	3	2	<b>4</b>	1
$S^2(C)$	1	3	5	<b>3</b>	2	4
$S^3(C)$	4	<b>1</b>	3	5	3	2
$S^4(C)$	2	4	1	<b>3</b>	5	3
$S^5(C)$	3	2	4	1	3	<b>5</b>

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$S^1(C)$	3	5	3	2	<b>4</b>	1
$S^2(C)$	1	3	5	<b>3</b>	2	4
$S^3(C)$	4	<b>1</b>	3	5	3	2
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## Remarks

- the fixed points are exactly the elements of the vector  $C$ ;

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$S^1(C)$	3	5	3	2	<b>4</b>	1
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$S^3(C)$	4	<b>1</b>	3	5	3	2
$S^4(C)$	2	4	1	<b>3</b>	5	3
$S^5(C)$	3	2	4	1	3	<b>5</b>

## Remarks

- the fixed points are exactly the elements of the vector  $C$ ;
- the sequence of the fixed points of a vector  $C(\pi) \in C_n$  is the vector  $C(\pi^{-1})$  belonging to  $C_n$ .

# Combinatorial characterizations of $C_n$

## Theorem

*A vector  $C \in C_n$  if and only if the zero vector is obtained from  $C$  by a finite sequence of switch operations.*

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*A vector  $C \in C_n$  if and only if the zero vector is obtained from  $C$  by a finite sequence of switch operations.*

## Example

<b>1</b>	<b>1</b>	1	0	2
0	<b>2</b>	<b>1</b>	0	2
0	0	<b>3</b>	<b>0</b>	2
0	0	4	<b>4</b>	<b>2</b>
0	0	<b>4</b>	<b>1</b>	0
0	0	0	0	0

# Combinatorial characterizations of $C_n$

## Definition

A *slide operation* is a switch operation applied to an index  $i = 0, \dots, n - 1$  such that  $c_{i+1} - c_i \geq 2$ .

If  $i = n - 1$  the slide operation transforms the vector  $(c_0, \dots, c_{n-1})$  into  $(c_{n-1} + 1, c_1, \dots, c_{n-2}, c_0 - 1)$ .

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## Theorem

Let  $C$  be a vector such that  $\sum_{i=0}^{n-1} c_i = k \cdot n$ . Then,  $C \in C_n$  if and only if it reduces to  $(k, \dots, k)$  in a finite sequence of slide operations.

# Combinatorial characterizations of $C_n$

## Example

Let  $C = (5, 3, 2, 4, 1, 3)$  be a vector, we have  $\sum_{i=0}^{n-1} c_i = 6 \cdot 3$ , we show a sequence of slide operations to obtain the vector  $(3, 3, 3, 3, 3, 3)$ :

5	3	<b>2</b>	<b>4</b>	1	3
<hr/>					
5	3	3	3	<b>1</b>	<b>3</b>
<hr/>					
<b>5</b>	3	3	3	2	<b>2</b>
<hr/>					
3	3	3	3	<b>2</b>	<b>4</b>
<hr/>					
3	3	3	3	3	3

## A new approach to Hall's Theorem

We believe that the result of the Theorem relating to the slide operation may be applied in order to give an efficient proof of Hall's Theorem.

Let  $C = \{1, 2, 2, 3, 4, 4, 5, 5, 5, 9\}$ ,  $\sum_{i=0}^9 c_i = 4 \cdot 10$ .

So the idea is to start from the vector  $(4, \dots, 4)$  and to find a rearrangement of  $C$ , considering that each element  $c_i$  has to move exactly  $|c_i - 4|$  positions.

### Example

$\{1, 2, 2, 3, 4, 4, 5, 5, 5, 9\}$

4 4 4 4 4 4 4 4 4 4

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### Example

$\{1, 2, 2, 3, 4, 4, 5, 5, 5, 9\}$

4 4 4 4 **4** 4 4 4 4 4

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### Example

$\{1, 2, 2, 3, 4, 4, 5, 5, 5, 9\}$

4 4 4 4 3 **5** 4 4 4 4

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### Example

$\{1, 2, 2, 3, 4, 4, 5, 5, 5, 9\}$

4 4 4 4 3 3 **6** 4 4 4

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### Example

$\{1, 2, 2, 3, 4, 4, 5, 5, 5, 9\}$

4 4 4 4 3 3 3 **7** 4 4

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Let  $C = \{1, 2, 2, 3, 4, 4, 5, 5, 5, 9\}$ ,  $\sum_{i=0}^9 c_i = 4 \cdot 10$ .

So the idea is to start from the vector  $(4, \dots, 4)$  and to find a rearrangement of  $C$ , considering that each element  $c_i$  has to move exactly  $|c_i - 4|$  positions.

### Example

$\{1, 2, 2, 3, 4, 4, 5, 5, 5, 9\}$

4 4 4 4 3 3 3 3 **8** 4

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So the idea is to start from the vector  $(4, \dots, 4)$  and to find a rearrangement of  $C$ , considering that each element  $c_i$  has to move exactly  $|c_i - 4|$  positions.

### Example

$\{1, 2, 2, 3, 4, 4, 5, 5, 5, 9\}$

4 4 4 4 3 3 3 3 3 **9**

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We believe that the result of the Theorem relating to the slide operation may be applied in order to give an efficient proof of Hall's Theorem.

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So the idea is to start from the vector  $(4, \dots, 4)$  and to find a rearrangement of  $C$ , considering that each element  $c_i$  has to move exactly  $|c_i - 4|$  positions.

### Example

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<b>4</b>	<b>4</b>	<b>1</b>	<b>5</b>	<b>5</b>	<b>2</b>	<b>2</b>	<b>5</b>	<b>3</b>	<b>9</b>

## Further Work

Let  $Q_n$  be the set of vectors of  $Z_n$  having zero-sum.

### Definition

Let  $\sigma, \pi$  be two permutations of  $S_n$ , the  $\sigma$ -transform of  $\pi$  is the vector

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### Remark

If we choose  $\sigma = id = (0, 1, \dots, n)$ , we have the ordinary identity transform  $C(\pi)$  of  $\pi$ .

## Further Work

### Remarks

- $Q_n$  is an abelian group, where the (commutative) sum  $\oplus$  of two elements is defined as the sum term by term of the two vectors, the neutral element of  $Q_n$  is  $(0, \dots, 0)$ , and the opposite of a generic element  $C = (c_0, \dots, c_{n-1}) \in Q_n$  is  $-C = (-c_0, \dots, -c_{n-1})$ ;

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- $Q_n$  is also closed under *scalar product*: given any  $h \in Z_n$ ,  
 $h \cdot C = (hc_0, \dots, hc_{n-1}) \in C_n$ ;
- $C_n$  is not a subgroup of  $Q_n$  since it is not closed under the sum  $\oplus$ .

## Further Work

The relation between  $C_n$  and  $Q_n$

### Proposition

*For every  $n$ , we have  $Q_n = C_n \ominus C_n$ , i.e. a vector  $Q$  belongs to  $Q_n$  if and only if it can be expressed as the difference of two elements  $C_1, C_2$  of  $C_n$ .*

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### Example

Let  $Q = (0, 1, 2, 4, 3) \in Q_5$  and  $Q \notin C_5$ .

$$\begin{array}{c|ccccc} \pi_1 & 1 & 3 & 0 & 4 & 2 \\ \pi_2 & 4 & 3 & 2 & 0 & 1 \\ \hline Q & 0 & 1 & 2 & 4 & 3 \end{array}$$

$$\begin{array}{c|ccccc} id & 0 & 1 & 2 & 3 & 4 \\ \pi_1 & 1 & 3 & 0 & 4 & 2 \\ \hline C_1 & 1 & 4 & 2 & 2 & 1 \end{array}$$

$$\begin{array}{c|ccccc} id & 0 & 1 & 2 & 3 & 4 \\ -\pi_2 & 1 & 2 & 3 & 0 & 4 \\ \hline C_2 & 1 & 3 & 5 & 3 & 3 \end{array}$$

Thus  $Q = C_1 \ominus C_2$ .

## Further Work

What about  $C_n \cap S_n$ ?

### Remarks

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- A permutation  $\pi \in C_n \cap S_n$  if and only if for all  $i, k$  we have  $\pi(i+k) - \pi(i) \neq k$ .
- The set  $C_n \cap S_n$  is closed under inversion (permutation).
- For every  $h$  which does not divide  $n$ , the vector  $h \cdot id$  and all its cyclic shifts belong to  $C_n \cap S_n$ .

With  $n = 5$ ,  $id = (0, 1, 2, 3, 4) \notin C_5 \cap S_5$ , while

$id \oplus id \oplus id = (0, 3, 1, 4, 2) \in C_5 \cap S_5$ .

There are elements in  $C_n \cap S_n$  which are not of type  $h \cdot id$  (or any of its cyclic shifts), for instance  $(0, 3, 1, 6, 5, 2, 4) \in C_7 \cap S_7$ .