

On Local Limit Theorems for $q(n)$ -Hypergeometric Series Discrete Distributions with $q(n) \rightarrow 1$ as $n \rightarrow \infty$

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Section 1: Introduction-Preliminaries

Definition of Basic (or q -) Hypergeometric Series for $0 < q < 1$

The q -analogue of a hypergeometric series

$${}_rF_s(a_1, \dots, a_r; b_1, \dots, b_s; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_r)_n z^n}{(b_1)_n \cdots (b_s)_n n!}$$

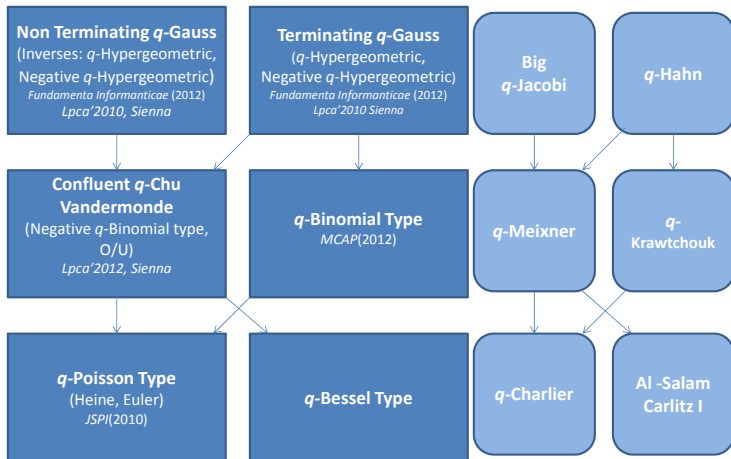
is

$${}_r\phi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, z) = \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r; q)_n z^n}{(b_1, \dots, b_s; q)_n (q; q)_n} [(-1)^n q^{\binom{n}{2}}]^{s-r+1},$$

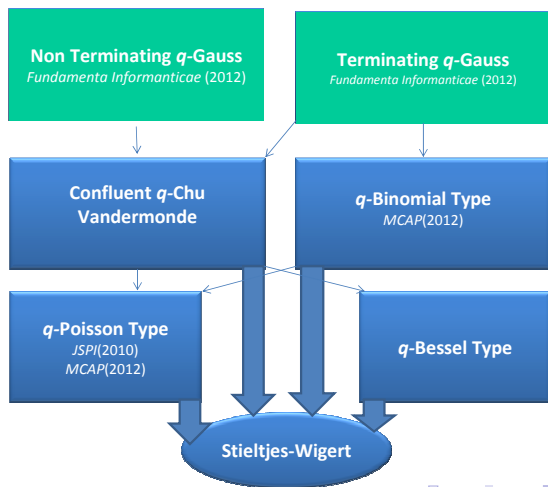
where

$$(a_1, \dots, a_r; q)_n = (a_1; q) \cdots (a_r; q)_n, \quad (a; q)_n = \prod_{j=1}^n (1 - aq^{j-1}), \quad (a; q)_0 = 1.$$

q -Hypergeometric Series Discrete Distributions and the Associated Scheme of Basic Hypergeometric Orthogonal Polynomials



What is the Continuous Limiting Behaviour of the q -Hypergeometric Series Discrete Distributions ?



Section 2: Main Results

Subsection 2.1: Continuous Limiting Behaviour of q -Binomial Distribution

Kemp and Kemp (1991) defined a q -analogue of the binomial distribution with probability function in the form

$$f_X(x) = \binom{n}{x}_q q^{\binom{x}{2}} \theta^x \prod_{j=1}^n (1 + \theta q^{j-1})^{-1}, \quad x = 0, 1, \dots, n, \quad (1)$$

where $\theta > 0$, $0 < q < 1$.

The mean and variance of the q -binomial distribution (1) are given respectively by

$$\begin{aligned} \mu &= \sum_{j=1}^n \frac{\theta q^{j-1}}{1 + \theta q^{j-1}} \\ \sigma^2 &= \sum_{j=1}^n \frac{\theta q^{j-1}}{(1 + \theta q^{j-1})^2}. \end{aligned} \quad (2)$$

- The q -binomial distribution (1), for q constant, has finite mean and variance when $n \rightarrow \infty$.
- The asymptotic normality in the sense of the DeMoivre-Laplace classical limit theorem can not be concluded, as in the case of the ordinary binomial distribution.
- Kyriakoussis and Vamvakari (2012), for q constant, established pointwise convergence of the q -binomial distribution (1) to a Stieltjes Wiger distribution as $n \rightarrow \infty$.
- Let $q = q(n)$ be a sequence of n with $q(n) \rightarrow 1$ as $n \rightarrow \infty$, emerging among others in statistical mechanics and analysis of probabilistic algorithms. How this assumption will affect the asymptotic behaviour of the q -binomial distribution (1)?

Theorem 1. The q -factorial numbers of order n , $[n]_q!$, where

(A) $q = q(n)$ with $q(n) \rightarrow 1$ as $n \rightarrow \infty$ and $q(n)^n = \Omega(1)$

or

(B) $q = q(n)$ with $q(n)^n = o(1)$

have the following asymptotic expansion for $n \rightarrow \infty$

$$\begin{aligned}
 [n]_q! &= (2\pi)^{1/2} q^{\binom{n}{2}} \exp(-g(r)) \left[rg'(r) + r^2 g''(r) \right]^{1/2} r^n \\
 &\cdot \left[1 + \sum_{k'=1}^N S_{k'}(r) (q^n(1-q))^{k'} + O\left((q^n(1-q))^{\frac{N+1}{2}} \right) \right]^{-1} \quad (3)
 \end{aligned}$$

where N is a positive integer, r is the real solution of the equation

$$rg'(r) = n$$

and

$$S_{k'}(r) = \frac{r^{k'}}{(2k')!} \sum_{j=1}^{2k'} B_{2k',j}(\alpha_1(r), \dots, \alpha_{2k'}(r)) \frac{\Gamma(j + k' + 1/2)}{\pi^{1/2}},$$

$$\alpha_{k'}(r) = \frac{\left[2^{-1} \left(1 + \frac{rg''(r)}{g'(r)} \right) \right]^{-k'/2-1} i^{k'+2} \sum_{\nu=1}^{k'+2} S(k'+2, \nu) r^\nu g^{(\nu)}(r)}{g'(r)(k'+1)(k'+2)r^{k'/2+1}} \quad (4)$$

with $B_{k',j}(\psi_1, \dots, \psi_{k'})$ the partial Bell polynomials, $S(k', j)$ the Stirling numbers of the second kind and $i^2 = -1$.

- The derived estimate for the q -factorial numbers of order n , is based on the analysis of the q -Exponential function

$$\begin{aligned}
 E_q((1-q)x) &= \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q; q)_n} (1-q)^n x^n \\
 &= \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{[n]_q!} x^n = \prod_{j=1}^{\infty} (1 + (1-q)xq^{j-1}) \quad (5)
 \end{aligned}$$

which is the ordinary generating function (g.f.) of the numbers $\frac{q^{\binom{n}{2}}}{[n]_q!}$, $n = 0, 1, 2, \dots$

Rewriting $E_q((1-q)x)$ as follows

$$E_q(x) = \exp(g(x)), \quad (6)$$

where

$$g(x) = \sum_{j=1}^{\infty} \log \left(1 + ((1-q)x)q^{j-1} \right). \quad (7)$$

Because of the large dominant singularities of the generating function $F_{m,q}(x)$, a well suited method for analyzing this is the *saddle point* method.

- We shall study the asymptotic behaviour of the q -factorial numbers of order n , $[n]_q!$, by expressing them via Cauchy's integral formula that gives the coefficients of a power series:

$$\frac{q^{\binom{n}{2}}}{[n]_q!} = \frac{1}{2\pi i} \int_{|x|=r} \frac{\exp(g(x))}{x^{n+1}} dx \quad (8)$$

where the contour of integration is taken to be a circle of radius r . This integral will be estimated with the saddle point method. The saddle point is defined by the equation $xg'(x) = n + 1$.

It turns out that it is convenient to switch to polar coordinates, setting $x = re^{i\theta}$. Then the original integral becomes

$$\frac{q^{\binom{n}{2}}}{[n]_q!} = \frac{\exp(g(r))}{r^n 2\pi} \int_{-\pi}^{\pi} \exp \left[g(re^{i\theta}) - g(r) - in\theta \right] d\theta. \quad (9)$$

In accordance with the saddle point method principles, we choose the radius r to be the solution of $rg'(r) = n$. Setting $G(\theta) = g(re^{i\theta}) - g(r) - in\theta$ with a Maclaurin series expansion about $\theta = 0$ we have

$$G(\theta) = -\phi^2 + \phi^2 \sum_{k'=1}^{\infty} \alpha_{k'}(r) \frac{(\psi\phi)^{k'}}{k'!} \quad (10)$$

where

$$\phi = \left[\frac{1}{2} (rg'(r) + r^2g''(r)) \right]^{1/2} \theta, \psi = [g'(r)]^{-\frac{1}{2}} \quad (11)$$

and

$$\alpha_{k'}(r) = \frac{\left[2^{-1} \left(1 + \frac{rg''(r)}{g'(r)} \right) \right]^{-k'/2-1} g^{(k'+2)}(re^{i\theta})|_{\theta=0}}{g'(r)(k'+1)(k'+2)r^{k'/2+1}}, \quad (12)$$

where

$$g^{(k'+2)}(re^{i\theta})|_{\theta=0} = \left(\frac{d}{d\theta} \right)^{k'+2} g(re^{i\theta})|_{\theta=0}.$$

The absence of a linear term in θ indicates a saddle point. The function $|e^{G(\theta)}|$ is unimodal with its peak at $\theta = 0$.

An estimation of the q -factorial numbers of order n , $[n]_q!$ with q defined by conditions (A) or (B) should naturally proceed by isolating separately small portions of the contour (corresponding to x near the real axis) as follows.

(A) For $q = q(n)$ with $q(n)^n = \Omega(1)$ we set

$$\begin{aligned} l_1 &= \frac{1}{\sqrt{2\pi}} \int_{-\delta}^{\delta} \exp[G(\theta)] d\theta, \\ l_2 &= \frac{1}{\sqrt{2\pi}} \int_{\delta}^{2\pi-\delta} \exp[G(\theta)] d\theta, \end{aligned} \quad (13)$$

and choose δ such that the following conditions are true (see [?]):

(C1) $n\delta^2 \rightarrow \infty$, that is $\delta \gg n^{-1/2}$

(C2) $n\delta^3 \rightarrow 0$, that is $\delta \ll n^{-1/3}$,

where “ \ll ” means “much smaller than”. A suitable choice for δ is $n^{-3/8}$.

As $|e^{G(\theta)}|$ decreases in $[\delta, \pi]$,

$$|e^{G(\theta)}| \leq |e^{G(\delta)}|, \delta \leq \theta \leq 2\pi - \delta. \quad (14)$$

We will show in the sequel that from (C1) and (C2) it follows that $e^{G(\delta)}$ is exponentially small, being dominated by a term of the form $e^{-O(n^{1/4})}$.

Indeed we have

$$\begin{aligned} G(\delta) &\sim -\frac{1}{2} \left(rg'(r) + r^2 g''(r) \right) \delta^2 \\ \text{or} \\ G(\delta) &\sim -\frac{1}{2} \left(rg'(r) + r^2 g''(r) \right) n^{-3/4}. \end{aligned} \quad (15)$$

But

$$rg'(r) + r^2 g''(r) \sim \frac{1}{\log q^{-1}} (q^{-n} - 1)$$

or

$$rg'(r) + r^2 g''(r) \sim \frac{(1-q)}{\log q^{-1}} (q^{n-1} + q^{n-2} + \dots + 1). \quad (16)$$

For $q = q(n)$ with $q(n)^n = \Omega(1)$ we get

$$G(\delta) = O\left(-\frac{1}{2} n^{1/4}\right). \quad (17)$$

From which we find that

$$|l_2| = O(e^{G(\delta)}) = O(e^{-\frac{1}{2} n^{1/4}}). \quad (18)$$

Thus, by (C1), δ has been taken large enough so that the central integral l_1 “captures” most of the contribution, while the remainder integral l_2 is exponentially small by (18).

We now turn to the precise evaluation of the central integral I_1 . We have

$$I_1 = \frac{1}{[rg'(r) + r^2g''(r)]^{1/2}} \frac{1}{\sqrt{\pi}} \int_{-\epsilon}^{\epsilon} \exp \left[-\phi^2 + \phi^2 \sum_{k'=1}^{\infty} \alpha_{k'}(r) \frac{(\psi\phi)^{k'}}{k'!} \right] d\phi \quad (19)$$

where

$$\epsilon = \left[\frac{1}{2} (rg'(r) + r^2g''(r)) \right]^{1/2} \delta. \quad (20)$$

Note that $\epsilon \rightarrow \infty$ as $n \rightarrow \infty$, since

$$\epsilon = n^{-3/8} \left[\frac{1}{2} (rg'(r) + r^2g''(r)) \right]^{1/2} = n^{1/8} \left[\frac{1}{2} \left(1 + \frac{rg''(r)}{g'(r)} \right) \right]^{1/2} > Cn^{1/8},$$

where C a positive constant.

(B) For $q = q(n)$ with $q(n)^n = o(1)$ we set

$$\begin{aligned} I_3 &= \frac{1}{\sqrt{2\pi}} \int_{-\delta}^{\delta} r^{-n} \exp[G(\theta)] d\theta, \\ I_4 &= \frac{1}{\sqrt{2\pi}} \int_{\delta}^{2\pi-\delta} r^{-n} \exp[G(\theta)] d\theta, \end{aligned} \quad (21)$$

and choose δ such that the conditions (C1) and (C2) are true. We suitably select $\delta = n^{-3/8}$.

As $|e^{G(\theta)}|$ decreases in $[\delta, \pi]$,

$$|e^{G(\theta)}| \leq |e^{G(\delta)}|, \delta \leq \theta \leq 2\pi - \delta. \quad (22)$$

We will now show that $e^{G(\delta)}$ is dominated by a term of the form $O(1)$. Indeed, from (C1), (C2), (15) and (16) it follows that

$$\exp(G(\delta)) \sim \exp\left(-\frac{1}{2} \frac{1}{\log q^{-1}} (1 - q^n) n^{-3/4}\right). \quad (23)$$

From which we get

$$|I_4| = O(r^{-n} e^{G(\delta)}) = O(q^{n^2} e^{-\frac{1}{2} \frac{1}{\log q^{-1}} (1 - q^n) n^{-3/4}}). \quad (24)$$

Thus, for $q = q(n)$ with $q(n)^n = o(1)$ the integral I_4 is negligibly small. We now turn to the precise evaluation of the central integral I_3 . Since

$$\begin{aligned}
 I_3 &= \frac{1}{\sqrt{2\pi}} \int_{-\delta}^{\delta} r^{-n} \exp[G(\theta)] d\theta \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r^{-n} \exp[G(\theta)] d\theta - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\delta} r^{-n} \exp[G(\theta)] d\theta \\
 &\quad - \frac{1}{\sqrt{2\pi}} \int_{\delta}^{\infty} r^{-n} \exp[G(\theta)] d\theta \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r^{-n} \exp[G(\theta)] d\theta - O(2q^{n^2} e^{-\frac{1}{2} \frac{1}{\log q^{-1}} (1-q^n)n^{-3/4}})
 \end{aligned}$$

we have

$$I_3 = \frac{r^{-n}}{[rg'(r) + r^2g''(r)]^{1/2}} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left[-\phi^2 + \phi^2 \sum_{k=1}^{\infty} \alpha_k(r) \frac{(\psi\phi)^k}{k!} \right] d\phi. \quad (25)$$

- We now unifiable proceed our proof for both conditions (A) and (B) and working analogously as in Kyriakoussis and Vamvakari (2012) we get our final estimation (3).

A q -Analogue of Stirling Type

In the previous theorem due to saddle point method principles, we have chosen the radius r of the derived asymptotic expansion (3) to be the solution of $rg'(r) = n$. By solving this saddle point equation we obtain the following.

Corollary 1. *The q -factorial numbers of order n , $[n]_q!$, where*

(A) $q = q(n)$ with $q(n) \rightarrow 1$ as $n \rightarrow \infty$ and $q(n)^n = \Omega(1)$

or

(B) $q = q(n)$ with $q(n)^n = o(1)$

have the following asymptotic expansion for $n \rightarrow \infty$

$$[n]_q! = \frac{(2\pi(1-q))^{1/2}}{(q \log q^{-1})^{1/2}} \frac{q^{\binom{n}{2}} q^{-n/2} [n]_{1/q}^{n+1/2}}{\prod_{j=1}^{\infty} (1 + q(q^{-n} - 1)q^{j-1})} (1 + O(q^n(1-q))). \quad (26)$$

Remark 1. Theorem 1(B) and Corollary 1(B) hold also for q constant.

Proposition 1(Kyriakoussis and Vamvakari(2012)). The q -mean and q -variance of the q -binomial distribution are given respectively by

$$\begin{aligned}\mu_q &= E([X]_{1/q}) \\ &= [n]_q \frac{\theta}{1 + \theta q^{n-1}}\end{aligned}$$

and (27)

$$\begin{aligned}\sigma_q^2 &= V([X]_{1/q}) \\ &= \frac{1-q}{q} [n]_q^2 \frac{\theta^2}{(1 + \theta q^{n-1})^2 (1 + \theta q^{n-2})} + [n]_q \frac{\theta}{(1 + \theta q^{n-1})(1 + \theta q^{n-2})}.\end{aligned}$$

Pointwise Convergence of the q -Binomial Distribution to the Stieltjes-Wigert Distribution

Theorem 2A(Kyriakoussis and Vamvakari(2012)). Let the p.f. of the q -binomial distribution be of the form

$$f_X(x) = \binom{n}{x}_q q^{\binom{x}{2}} \theta^x \prod_{j=1}^n (1 + \theta q^{j-1})^{-1}, \quad x = 0, 1, \dots, n,$$

where $\theta = \theta_n$, $n = 0, 1, 2, \dots$ such that $\theta_n = q^{-an}$ with $0 < a < 1$ constant. Then, for $n \rightarrow \infty$, the q -binomial distribution is approximated by a deformed standardized continuous Stieltjes-Wigert distribution as follows

$$\begin{aligned} f_X(x) &\cong \frac{q^{1/8}(\log q^{-1})^{1/2}}{(2\pi)^{1/2}} \left(q^{-3/2}(1-q)^{1/2} \frac{[x]_{1/q} - \mu_q}{\sigma_q} + q^{-1} \right)^{1/2} \\ &\cdot \exp \left(\frac{1}{2 \log q} \log^2 \left(q^{-3/2}(1-q)^{1/2} \frac{[x]_{1/q} - \mu_q}{\sigma_q} + q^{-1} \right) \right), \\ &x \geq 0. \end{aligned} \tag{28}$$

Fundamental Steps to Establish Pointwise Convergence in a q -Analogue sense of the DeMoivre-Laplace Classical Limit Theorem

- 1 Substitution of the q -analogue of Stirling type (26) to the q -binomial distribution (1)

- 2 Transition to variable $[X]_{1/q} = \frac{1-q^{-X}}{1-q^{-1}}$

- 3 Perform q -standardization

$$Z = \frac{[X]_{1/q} - \mu_q}{\sigma_q}$$

with μ_q and σ_q given in (27)

- 4 Proceed with all needed asymptotics and algebraic manipulations

Local Limit Theorems for the $q(n)$ -Binomial Distribution with $q(n) \rightarrow 1$ as $n \rightarrow \infty$

Theorem 2B. Let the p.f. of the q -binomial distribution be of the form

$$f_X(x) = \binom{n}{x}_q q^{\binom{x}{2}} \theta^x \prod_{j=1}^n (1 + \theta q^{j-1})^{-1}, \quad x = 0, 1, \dots, n,$$

where $\theta = \theta_n$, $n = 0, 1, 2, \dots$ such that $\theta_n = q^{-\alpha n}$ with $0 < a < 1$ constant and $q = q(n)$ with $q(n) \rightarrow 1$ as $n \rightarrow \infty$ and $q(n)^n = o(1)$. Then, for $n \rightarrow \infty$, the q -binomial distribution is approximated by a deformed standardized Gauss distribution as follows

$$f_X(x) \cong \frac{(1-q)\mu_q}{\sigma_q(2\pi)^{1/2}} \exp\left(-\frac{1}{2} \left(\frac{[X]_{1/q} - \mu_q}{\sigma_q}\right)^2\right), \quad x \geq 0 \quad (29)$$

with

$$\mu_q \cong \frac{q^{-\alpha n}}{1-q} \quad \text{and} \quad \sigma_q^2 \cong \frac{q^{-\alpha n}}{1-q}. \quad (30)$$

Remark 2. Possible realizations of the sequence $q(n)$, $n = 0, 1, 2, \dots$ considered in the above theorem 2 are among others the next two ones

$$q(n) = 1 - \frac{1}{\ln(n)} \quad \text{or} \quad q(n) = 1 - \frac{1}{n^{1/2}}.$$

- ④ Substitution of the q -analogue of Stirling type (26), for $q = q(n)$ with $q(n) \rightarrow 1$ and $q(n)^n = o(1)$, to the q -binomial distribution (1)

$$f_X(x) \cong \frac{(q \log q^{-1})^{1/2}}{(2\pi(1-q))^{1/2}} \frac{\theta_n^x}{(1-q)^x} \frac{\prod_{j=1}^{\infty} (1 + q(q^{-x} - 1)q^{j-1})}{\prod_{j=1}^{\infty} (1 + \theta_n q^{j-1}) q^{-x/2} [x]_{1/q}^{x+1/2}}. \quad (31)$$

- ② Transition to variable $[X]_{\frac{1}{q}} = \frac{1-q^{-X}}{1-q^{-1}}$
- ③ Perform q -standardization

$$Z = \frac{[X]_{1/q} - \mu_q}{\sigma_q}$$

with μ_q and σ_q given asymptotically by (30)

- ④ Proceed with all needed asymptotics and algebraic manipulations

- $[x]_{1/q} \cong \frac{\theta_n}{(1-q)} \left(\left(\frac{\theta_n}{(1-q)} \right)^{-1/2} z + 1 \right)$
- $q^{-x} \cong (q^{-1} - 1) \left(\frac{\theta_n}{(1-q)} \right) \left(\left(\frac{\theta_n}{(1-q)} \right)^{-1/2} z + 1 \right) \quad (32)$

- $x \cong \frac{1}{\log q^{-1}} \log \left(\left(\frac{\theta_n}{q} \right) \left(\left(\frac{\theta_n}{(1-q)} \right)^{-1/2} z + 1 \right) \right)$ (33)

- $[x]_{1/q}^x \cong \left(\frac{\theta_n}{1-q} \right)^x$
 $\cdot \exp \left(\frac{1}{\log q^{-1}} \log \left(\frac{\theta_n}{q} \left(\left(\frac{\theta_n}{(1-q)} \right)^{1/2} z + 1 \right) \right) \log \left(\left(\frac{\theta_n}{(1-q)} \right)^{-1/2} z + 1 \right) \right)$ (34)

- $\prod_{j=1}^{\infty} \left(1 + \theta_n \left(\left(\frac{\theta_n}{(1-q)} \right)^{-1/2} z + 1 \right) q^{j-1} \right)$
 $= \exp \left(\sum_{j=1}^{\infty} \log \left(1 + \theta_n \left(\left(\frac{\theta_n}{(1-q)} \right)^{-1/2} z + 1 \right) q^{j-1} \right) \right),$ (35)

where

$$\begin{aligned}
 & \sum_{j=1}^{\infty} \log \left(1 + \theta_n \left(\left(\frac{\theta_n}{(1-q)} \right)^{-1/2} z + 1 \right) q^{j-1} \right) \\
 &= \frac{1}{2 \log q^{-1}} \log^2 \left(\theta_n \left(\left(\frac{\theta_n}{(1-q)} \right)^{-1/2} z + 1 \right) \right) \\
 &+ \text{Li}_2 \left(\frac{\theta_n \left(\left(\frac{\theta_n}{(1-q)} \right)^{-1/2} z + 1 \right)}{\theta_n \left(\left(\frac{\theta_n}{(1-q)} \right)^{1/2} z + 1 \right) + 1} \right) \\
 &+ \frac{1}{2} \log \left(1 + \theta_n \left(\left(\frac{\theta_n}{(1-q)} \right)^{-1/2} z + 1 \right) \right) \\
 &+ \frac{\beta_2 \log q}{2} \frac{\theta_n \left(\left(\frac{\theta_n}{(1-q)} \right)^{-1/2} z + 1 \right)}{1 + \theta_n \left(\left(\frac{\theta_n}{(1-q)} \right)^{-1/2} z + 1 \right)} + O \left(\frac{1}{\theta_n} \right). \quad (36)
 \end{aligned}$$

- $$\prod_{j=1}^{\infty} (1 + \theta_n q^{j-1}) = \exp \left(\sum_{j=1}^{\infty} \log (1 + \theta_n q^{j-1}) \right),$$

where

$$\begin{aligned} \sum_{j=1}^{\infty} \log (1 + \theta_n q^{j-1}) &= \frac{1}{2 \log q^{-1}} \log^2 (\theta_n) + \text{Li}_2 \left(\frac{\theta_n}{\theta_n + 1} \right) + \frac{1}{2} \log (1 + \theta_n) \\ &+ \frac{\beta_2 \log q}{2} \frac{\theta_n}{1 + \theta_n} + O(\theta_n^{-1}) \end{aligned} \quad (37)$$

Applying all the previous the estimations (32)-(37) to the approximation (31), carrying out all the necessary manipulations and by the assumptions $\theta_n = q^{-\alpha n}$ with $0 < a < 1$ constant and $q = q(n)$ with $q(n) \rightarrow 1$ as $n \rightarrow \infty$ and $q(n)^n = o(1)$, we derive our final asymptotic (29).

Theorem 2C. Let the p.f. of the q -binomial distribution be of the form

$$f_X(x) = \binom{n}{x}_q q^{\binom{x}{2}} \theta^x \prod_{j=1}^n (1 + \theta q^{j-1})^{-1}, \quad x = 0, 1, \dots, n,$$

where $\theta = \theta_n$, $n = 0, 1, 2, \dots$ such that $\theta_n \rightarrow \infty$, as $n \rightarrow \infty$ and $q = q(n)$ with $q(n) = 1 - \frac{\beta}{n}$, $0 < \beta \leq 1$. Then, for $n \rightarrow \infty$, the q -binomial distribution is approximated by a deformed Gauss distribution as follows

$$f_X(x) \cong \frac{((1-q)\mu_q + 1)}{\frac{\sigma_q}{\lambda_\beta^{1/2}} (2\pi)^{1/2}} \exp\left(-\frac{1}{2} \left(\frac{[x]_{1/q} - \mu_q}{\frac{\sigma_q}{\lambda_\beta^{1/2}}}\right)^2\right), \quad x \geq 0, \quad (38)$$

where

$$\lambda_\beta = \frac{(2 - \beta) \exp(\beta) - (1 - \beta)(\exp(\beta) - 1)}{\exp(\beta)} \quad (39)$$

and with

$$\mu_q \cong \frac{\exp(\beta) - 1}{\beta \exp(\beta)} n \quad \text{and} \quad \sigma_q^2 \cong \frac{\exp(\beta) - 1}{\beta \exp(\beta)} n.$$

Remark 3. For $\beta = 1$ and $q = q(n) = 1 - \frac{1}{n}$ the approximation of the above theorem 3 is a deformed standardized Gauss distribution.

Section 2: Main Results

Subsection 2.2: Continuous Limiting Behaviour of Confluent q -Chu-Vandermonde Distributions

Family of Confluent q -Chu-Vandermonde Hypergeometric Series Distributions

- Confluent q -Chu-Vandermonde sum

$$\begin{aligned} {}_1\phi_1(b; c; q, c/b) &= \sum_{n=0}^{\infty} \frac{(b; q)_n}{(c; q)_n (q; q)_n} (-c/b)^n q^{\binom{n}{2}} \\ &= \frac{(c/b; q)_{\infty}}{(c; q)_{\infty}}, 0 < q < 1, \end{aligned} \quad (40)$$
$$(a; q)_x = \prod_{j=1}^x (1 - aq^{j-1}), x = 0, 1, 2, \dots$$

- Rising up family of confluent q -Chu-Vandermonde series distributions for suitable values of c and b

$$p(x) = P[X = x] = f_X(x) = \frac{\frac{(b; q)_x}{(c; q)_x (q; q)_x} q^{\binom{x}{2}} (-c/b)^x}{\frac{(c/b; q)_{\infty}}{(c; q)_{\infty}}}, x = 0, 1, \dots \quad (41)$$

(see Kemp(2005)).

Members of the Family of Confluent q -Chu-Vandermonde Series Distributions

Confluent q -Chu-Vandermonde Distributions	Parameters b and c	Support
q -CCV-I	$b = -h, h > 0, 0 < c < 1$	$x = 0, 1, 2, \dots$
q -CCV-II	$0 < b < 1, c = -\eta, \eta > 0$	$x = 0, 1, 2, \dots$
q -CCV-III	$b = q^{-n}, n = 0, 1, \dots, 0 < c < 1$	$x = 0, 1, \dots, n$

q -Factorial and Factorial Moments of the Confluent q -Chu-Vandermonde Distributions

- **Proposition 2.** The r -th order q -factorial moments of the family of confluent q -Chu-Vandermonde distributions are given by

$$E([X]_{r,1/q}) = \frac{(b; q)_r}{(1-q)^r} \left(-\frac{c}{b}\right)^r, \quad r = 1, 2, \dots \quad (42)$$

- **Proposition 3.** The q -mean and q -variance of the family of confluent q -Chu-Vandermonde distributions are given respectively by

$$\begin{aligned} \mu_q &= E([X]_{1/q}) = -\frac{c}{b} \frac{1-b}{1-q} \\ \sigma_q^2 &= V([X]_{1/q}) = \left(-\frac{c}{b}\right)^2 \frac{1-b}{q(1-q)} - \frac{c}{b} \frac{1-b}{1-q}. \end{aligned} \quad (43)$$

- **Proposition 4.** The descending factorial k -th order moments of the r.v. X of the family of q -Chu-Vandermonde distributions are given by

$$E((X)_k) = \frac{k!}{(c/b; q)_\infty} \sum_{r=k}^{\infty} \frac{s_q(r, k)(cq^r; q)_\infty}{(q-1)^{k-r} [r]_{1/q}!} E([X]_{r,1/q}) {}_1\phi_1(bq^r; cq^r; q, cq^r/b). \quad (44)$$

Pointwise Convergence of a family of confluent q -Chu-Vandermonde distributions to Stieltjes-Wigert Distribution

Theorem 3A. Let the p.f. of the family of confluent q -Chu-Vandermonde distributions be of the form

$$f_X(x) = \frac{\frac{(b;q)_x}{(c;q)_x(q;q)_x} q^{\binom{x}{2}} (-c/b)^x}{\frac{(c/b;q)_\infty}{(c;q)_\infty}}, \quad x = 0, 1, \dots \quad (45)$$

where $b = b_n$, $c = c_n$, $n = 0, 1, 2, \dots$, such that $b_n = o(1)$ and $-c_n/b_n \rightarrow \infty$, as $n \rightarrow \infty$. Then, for $n \rightarrow \infty$, the p.f. $f_X(x)$, $x = 0, 1, 2, \dots$ is approximated by a deformed standardized continuous Stieltjes-Wigert distribution as follows

$$\begin{aligned} f_X(x) &\cong \frac{q^{1/8} (\log q^{-1})^{1/2}}{(2\pi)^{1/2}} \left(q^{-3/2} (1-q)^{1/2} \frac{[x]_{1/q} - \mu_q}{\sigma_q} + q^{-1} \right)^{1/2} \\ &\cdot \exp \left(\frac{1}{2 \log q} \log^2 \left(q^{-3/2} (1-q)^{1/2} \frac{[x]_{1/q} - \mu_q}{\sigma_q} + q^{-1} \right) \right), \\ &x \geq 0. \end{aligned} \quad (46)$$

- 1 Substitution of the q -analogue of Stirling type (26), for q constant, to the p.f. of the family confluent q -Chu Vandermonde distribution (41)

$$f_X(x) \cong \frac{(q \log q^{-1})^{1/2} (-c_n/b_n)^x \prod_{j=1}^{\infty} (1 + q(q^{-x} - 1)q^{j-1})(c_n; q)_{\infty}}{(2\pi(1-q))^{1/2} (1-q)^x q^{-x/2} [x]_{1/q}^{x+1/2} (c_n; q)_x (c_n/b_n; q)_{\infty}} \quad (47)$$

- 2 Transition to variable $[X]_{1/q} = \frac{1-q^{-x}}{1-q^{-1}}$
- 3 Perform q -standardization

$$Z = \frac{[X]_{1/q} - \mu_q}{\sigma_q}$$

with μ_q and σ_q given by (43).

- 4 Proceed with all needed asymptotics

$$\begin{aligned}
 & \bullet [x]_{1/q} \cong -\frac{c_n}{b_n} \frac{q}{1-q} (q^{-3/2}(1-q)^{1/2}z + q^{-1}) \\
 & \bullet q^{-x} \cong -\frac{c_n}{b_n} \left(q^{-3/2}(1-q)z + q^{-1} \right) \\
 & \bullet x \cong \frac{1}{\log q^{-1}} \log \left(-\frac{c_n}{b_n} \left(q^{-3/2}(1-q)z + q^{-1} \right) \right).
 \end{aligned} \quad (48)$$

- $$[x]_{1/q}^x \cong \left(-\frac{c_n}{b_n}\right)^x \left(\frac{1}{1-q}\right)^x$$

$$\cdot \exp\left(\frac{1}{\log q^{-1}} \log\left(-\frac{c_n}{b_n} \left(\frac{1-q}{q^{3/2}}z + q^{-1}\right)\right) \log\left(q \left(\frac{1-q}{q^{3/2}}z + q^{-1}\right)\right)\right)$$
(49)

- $$\prod_{j=1}^{\infty} \left(1 - \frac{c_n}{b_n} \left(\frac{(1-q)^{1/2}}{q^{3/2}}z + q^{-1}\right) q^{j-1}\right)$$

$$= \exp\left(\sum_{j=1}^{\infty} \log\left(1 - \frac{c_n}{b_n} \left(\frac{(1-q)^{1/2}}{q^{3/2}}z + q^{-1}\right) q^{j-1}\right)\right),$$
(50)

where

$$\begin{aligned}
 & \sum_{j=1}^{\infty} \log \left(1 - \frac{c_n}{b_n} \left(q^{-3/2} (1-q)^{1/2} z + q^{-1} \right) q^{j-1} \right) \\
 &= \frac{1}{2 \log q^{-1}} \log^2 \left(-\frac{c_n}{b_n} \left(q^{-3/2} (1-q)^{1/2} z + q^{-1} \right) \right) \\
 &+ \operatorname{Li}_2 \left(\frac{-\frac{c_n}{b_n} \left(q^{-3/2} (1-q)^{1/2} z + q^{-1} \right)}{-\frac{c_n}{b_n} \left(q^{-3/2} (1-q)^{1/2} z + q^{-1} \right) + 1} \right) \\
 &+ \frac{1}{2} \log \left(1 - \frac{c_n}{b_n} \left(q^{-3/2} (1-q)^{1/2} z + q^{-1} \right) \right) \\
 &+ \frac{\beta_2 \log q}{2} \frac{\left(-\frac{c_n}{b_n} \right) \left(q^{-3/2} (1-q)^{1/2} z + q^{-1} \right)}{1 - \frac{c_n}{b_n} \left(q^{-3/2} (1-q)^{1/2} z + q^{-1} \right)} + O \left(-\frac{b_n}{c_n} \right) \quad (51)
 \end{aligned}$$

$$\bullet \prod_{j=1}^{\infty} \left(1 - \frac{c_n}{b_n} q^{j-1}\right) = \exp \left(\sum_{j=1}^{\infty} \log \left(1 - \frac{c_n}{b_n} q^{j-1}\right) \right), \quad (52)$$

where

$$\begin{aligned} \sum_{j=1}^{\infty} \log \left(1 - \frac{c_n}{b_n} q^{j-1}\right) &= \frac{1}{2 \log q^{-1}} \log^2 \left(-\frac{c}{b_n}\right) + \text{Li}_2 \left(\frac{-\frac{c}{b_n}}{-\frac{c}{b_n} + 1}\right) \\ &+ \frac{1}{2} \log \left(1 - \frac{c}{b_n}\right) \\ &+ \frac{\beta_2 \log q}{2} \frac{\left(-\frac{c}{b_n}\right)}{1 - \frac{c}{b_n}} + O\left(-\frac{b_n}{c}\right) \end{aligned} \quad (53)$$

$$\begin{aligned}
 \bullet \quad (c_n; q)_\infty / (c_n; q)_x &= (c_n q^x; q)_\infty = \prod_{j=1}^{\infty} (1 - c_n q^x q^{j-1}) \\
 &= \exp \left(\sum_{j=1}^{\infty} \log \left(1 + b_n \left(q^{-3/2} (1 - q)^{1/2} z + q^{-1} \right)^{-1} q^{j-1} \right) \right), \quad (54)
 \end{aligned}$$

where

$$\begin{aligned}
 &\sum_{j=1}^{\infty} \log \left(1 + b_n \left(q^{-3/2} (1 - q)^{1/2} z + q^{-1} \right)^{-1} q^{j-1} \right) \\
 &= \frac{1}{2 \log q^{-1}} \log^2 \left(b_n \left(q^{-3/2} (1 - q)^{1/2} z + q^{-1} \right)^{-1} \right) \\
 &\quad + \text{Li}_2 \left(\frac{b_n \left(q^{-3/2} (1 - q)^{1/2} z + q^{-1} \right)^{-1}}{b_n \left(q^{-3/2} (1 - q)^{1/2} z + q^{-1} \right)^{-1} + 1} \right) \\
 &\quad + \frac{1}{2} \log \left(1 + b_n \left(q^{-3/2} (1 - q)^{1/2} z + \frac{1}{q} \right)^{-1} \right)
 \end{aligned}$$

$$+ \frac{\beta_2 \log q}{2} \frac{b_n \left(q^{-3/2} (1-q)^{1/2} z + q^{-1} \right)^{-1}}{1 + b_n \left(q^{-3/2} (1-q)^{1/2} z + q^{-1} \right)^{-1}} + O(b_n) \quad (55)$$

Applying all the previous the estimations (48)-(55) to the approximation (47), carrying out all the necessary manipulations and by the assumptions $b_n = o(1)$ and $-c_n/b_n \rightarrow \infty$, as $n \rightarrow \infty$, we derive our final asymptotic (46).

- Under the assumptions of the theorem 3A confluent q -Chu-Vandermonde distributions I and II (41) converge to a continuous Stieltjes Wigert distribution.
- The confluent q -Chu-Vandermonde distribution III (41) does not have the same continuous limiting behaviour since the assumption $b_n = o(1)$ of theorem 3A does not hold.

Local Limit Theorems for the Confluent $q(n)$ -Chu-Vandermonde Hypergeometric Series Discrete Distributions I and II with $q(n) \rightarrow 1$ as $n \rightarrow \infty$

Theorem 3B. Let the p.f. of the family of confluent q -Chu-Vandermonde distributions be of the form

$$f_X(x) = \frac{\frac{(b;q)_x}{(c;q)_x(q;q)_x} q^{\binom{x}{2}} (-c/b)^x}{\frac{(c/b;q)_\infty}{(c;q)_\infty}}, \quad x = 0, 1, \dots \quad (56)$$

where $b = b_n$, $c = c_n$, $n = 0, 1, 2, \dots$, such that $b_n = o(1)$ and $-c_n/b_n \rightarrow \infty$, as $n \rightarrow \infty$ and $q = q(n)$ with $q(n) \rightarrow 1$ as $n \rightarrow \infty$ and $q(n)^n = o(1)$. Then, for $n \rightarrow \infty$, the p.f. $f_X(x)$, $x = 0, 1, 2, \dots$ is approximated by a deformed standardized Gauss distribution as follows

$$f_X(x) \cong \frac{(1-q)\mu_q}{\sigma_q(2\pi)^{1/2}} \exp\left(-\frac{1}{2} \left(\frac{[x]_{1/q} - \mu_q}{\sigma_q}\right)^2\right), \quad x \geq 0 \quad (57)$$

with

$$\mu_q \cong -\frac{c_n}{b_n(1-q)} \quad \text{and} \quad \sigma_q^2 \cong -\frac{c_n}{b_n(1-q)}. \quad (58)$$

Remark 4. The above approximation (57) holds also for $-\frac{c_n}{b_n} = \Theta(q^{-an})$ with $0 < a < 1$.

- ① Substitution of the q -analogue of Stirling type (26), for $q = q(n)$ with $q(n) \rightarrow 1$ as $n \rightarrow \infty$ and $q(n)^n = o(1)$, to the p.f. of the family confluent q -Chu Vandermonde distribution (41)

$$f_X(x) \cong \frac{(q \log q^{-1})^{1/2}}{(2\pi(1-q))^{1/2}} \frac{(-c_n/b_n)^x}{(1-q)^x} \frac{\prod_{j=1}^{\infty} (1 + q(q^{-x} - 1)q^{j-1})(c_n; q)_{\infty}}{q^{-x/2} [X]_{1/q}^{x+1/2} (c_n; q)_x (c_n/b_n; q)_{\infty}}. \quad (59)$$

- ② Transition to variable $[X]_{\frac{1}{q}} = \frac{1-q^{-X}}{1-q^{-1}}$
 ③ Perform q -standardization

$$Z = \frac{[X]_{1/q} - \mu_q}{\sigma_q}$$

with μ_q and σ_q given by (58).

- ④ Proceed with all needed asymptotics

$$\begin{aligned} \bullet [X]_{1/q} &\cong -\frac{c_n}{b_n(1-q)} \left(\left(-\frac{c_n}{b_n(1-q)} \right)^{-1/2} z + 1 \right) \\ \bullet q^{-x} &\cong \left(-\frac{c_n}{qb_n} \right) \left(\left(-\frac{c_n}{b_n(1-q)} \right)^{-1/2} z + 1 \right) \end{aligned} \quad (60)$$

$$\bullet \quad x \cong \frac{1}{\log q^{-1}} \log \left(\left(-\frac{c_n}{qb_n} \right) \left(\left(-\frac{c_n}{b_n(1-q)} \right)^{-1/2} z + 1 \right) \right) \quad (61)$$

$$\bullet \quad [X]_{1/q}^x \cong \left(-\frac{c_n}{b_n} \right)^x \left(\frac{1}{1-q} \right)^x$$

$$\cdot \exp \left(\frac{1}{\log q^{-1}} \log \left(\left(-\frac{c_n}{qb_n} \right) \left(\left(-\frac{c_n}{b_n(1-q)} \right)^{1/2} z + 1 \right) \right) \right)$$

$$\cdot \log \left(\left(-\frac{c_n}{b_n(1-q)} \right)^{-1/2} z + 1 \right) \quad (62)$$

$$\bullet \quad \prod_{j=1}^{\infty} \left(1 - \frac{c_n}{b_n} \left(\left(-\frac{c_n}{b_n(1-q)} \right)^{-1/2} z + 1 \right) q^{j-1} \right)$$

$$= \exp \left(\sum_{j=1}^{\infty} \log \left(1 - \frac{c_n}{b_n} \left(\left(-\frac{c_n}{b_n(1-q)} \right)^{-1/2} z + 1 \right) q^{j-1} \right) \right), \quad (63)$$

where

$$\begin{aligned}
 & \sum_{j=1}^{\infty} \log \left(1 - \frac{c_n}{b_n} \left(\left(-\frac{c_n}{b_n(1-q)} \right)^{-1/2} z + 1 \right) q^{j-1} \right) \\
 &= \frac{1}{2 \log q^{-1}} \log^2 \left(-\frac{c_n}{b_n} \left(\left(-\frac{c_n}{b_n(1-q)} \right)^{-1/2} z + 1 \right) \right) \\
 &+ \operatorname{Li}_2 \left(\frac{-\frac{c_n}{b_n} \left(\left(-\frac{c_n}{b_n(1-q)} \right)^{-1/2} z + 1 \right)}{-\frac{c_n}{b_n} \left(\left(-\frac{c_n}{b_n(1-q)} \right)^{1/2} z + 1 \right) + 1} \right) \\
 &+ \frac{1}{2} \log \left(1 - \frac{c_n}{b_n} \left(\left(-\frac{c_n}{b_n(1-q)} \right)^{-1/2} z + 1 \right) \right) \\
 &+ \frac{\beta_2 \log q}{2} \frac{\left(-\frac{c_n}{b_n} \right) \left(\left(-\frac{c_n}{b_n(1-q)} \right)^{-1/2} z + 1 \right)}{1 - \frac{c_n}{b_n} \left(\left(-\frac{c_n}{b_n(1-q)} \right)^{-1/2} z + 1 \right)} + O \left(-\frac{b_n}{c_n} \right). \quad (64)
 \end{aligned}$$

$$\begin{aligned}
 \bullet \quad (c_n; q)_\infty / (c_n; q)_x &= (c_n q^x; q)_\infty = \prod_{j=1}^{\infty} (1 - c_n q^x q^{j-1}) \\
 &= \exp \left(\sum_{j=1}^{\infty} \log \left(1 + qb_n \left(\left(-\frac{c_n}{b_n(1-q)} \right)^{-1/2} z + 1 \right)^{-1} q^{j-1} \right) \right),
 \end{aligned} \tag{65}$$

where

$$\begin{aligned}
 &\sum_{j=1}^{\infty} \log \left(1 + qb_n \left(\left(-\frac{c_n}{b_n(1-q)} \right)^{-1/2} z + 1 \right)^{-1} q^{j-1} \right) \\
 &= \frac{1}{2 \log q^{-1}} \log^2 \left(qb_n \left(\left(-\frac{c_n}{b_n(1-q)} \right)^{-1/2} z + 1 \right)^{-1} \right) \\
 &\quad + \text{Li}_2 \left(\frac{qb_n \left(\left(-\frac{c_n}{b_n(1-q)} \right)^{-1/2} z + 1 \right)^{-1}}{qb_n \left(\left(-\frac{c_n}{b_n(1-q)} \right)^{-1/2} z + 1 \right)^{-1} + 1} \right)
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \log \left(1 + qb_n \left(\left(-\frac{c_n}{b_n(1-q)} \right)^{-1/2} z + 1 \right)^{-1} \right) \\
& + \frac{\beta_2 \log q}{2} \frac{qb_n \left(\left(-\frac{c_n}{b_n(1-q)} \right)^{-1/2} z + 1 \right)^{-1}}{1 + qb_n \left(\left(-\frac{c_n}{b_n(1-q)} \right)^{-1/2} z + 1 \right)^{-1}} + O(b_n). \quad (66)
\end{aligned}$$

Applying all the previous the estimations (60)-(66) to the approximation (59), carrying out all the necessary Taylor-Maclaurin expansions and manipulations and by the assumptions $b_n = o(1)$ and $-c_n/b_n \rightarrow \infty$, as $n \rightarrow \infty$ and $q = q(n)$ with $q(n) \rightarrow 1$ as $n \rightarrow \infty$ and $q(n)^n = o(1)$ we derive our final asymptotic (57).

Theorem 3C. Let the p.f. of the family of confluent q -Chu-Vandermonde distributions be of the form

$$f_X(x) = \frac{\frac{(b;q)_x}{(c;q)_x(q;q)_x} q^{\binom{x}{2}} (-c/b)^x}{\frac{(c/b;q)_\infty}{(c;q)_\infty}}, \quad x = 0, 1, \dots \quad (67)$$

where $b = b_n$, $c = c_n$, $n = 0, 1, 2, \dots$, such that $b_n = o(1)$ and $q = q(n)$ with $q(n) = 1 - \frac{\beta}{n}$, $0 < \beta \leq 1$. Then, for $n \rightarrow \infty$, the p.f. $f_X(x)$, $x = 0, 1, 2, \dots$ is approximated by a deformed standardized Gauss distribution as follows

$$f_X(x) \cong \frac{(1-q)\mu_q}{\sigma_q(2\pi)^{1/2}} \exp\left(-\frac{1}{2} \left(\frac{[x]_{1/q} - \mu_q}{\sigma_q}\right)^2\right), \quad x \geq 0 \quad (68)$$

with

$$\mu_q \cong -\frac{cn}{b\beta} \quad \text{and} \quad \sigma_q^2 \cong -\frac{cn}{b\beta}.$$

Remark 5. For $-\frac{c_n}{b_n} = \Theta(q^{-an})$ with $0 < a < 1$ and $q(n) = 1 - \frac{\beta}{n}$, $0 < \beta \leq 1$, the p.f. of the family of confluent q -Chu-Vandermonde distributions is approximated by a deformed Gauss distribution as follows

$$f_X(x) \cong \frac{((1-q)\mu_q + 1)}{\lambda_{a,\beta}^{1/2} (2\pi)^{1/2}} \exp\left(-\frac{1}{2} \left(\frac{[x]_{1/q} - \mu_q}{\lambda_{a,\beta}^{1/2}}\right)^2\right), \quad x \geq 0, \quad (69)$$

where

$$\lambda_{a,\beta} = \frac{1}{\exp(a\beta) + c} \left[\frac{\exp(a\beta) + c - 1}{\exp(a\beta) + c + 1} + \log \frac{\exp(a\beta) + c + 1}{\exp(a\beta) + c} \right] \quad (70)$$

and with

$$\mu_q \cong \frac{\exp(a\beta) + c}{\beta} n \quad \text{and} \quad \sigma_q^2 \cong \frac{\exp(a\beta) + c}{\beta} n.$$

Remark 6. Under the assumptions of the above two theorems 4 and 5 the probability functions I and II of the table 1 have the asymptotic approximations (57) or (68). Note that we do not have the same conclusion for the p.f. III of the table 1 since the assumption $b_n = o(1)$ does not hold. A continuous analogue for the p.f. III established by Berg and Valent (1994). But there still remains an open problem to find the asymptotic behaviour in the sense of pointwise convergence of the confluent q -Chu-Vandermonde III, for both q constant and $q = q(n)$.

- **A modified q -Bessel distribution:** A discrete approximation of the confluent q -Chu-Vandermonde I, for $b_n = -h_n = o(1)$, $h_n > 0$ and $c_n = c$ constant with $0 < c < 1$, is a modified q -Bessel distribution with p.f.

$$p_{MB}(x) = \frac{q^{\binom{x}{2}} (c/h_n)^x}{\frac{(c;q)_x (q;q)_x}{(-c/h_n; q)_\infty (c;q)_\infty}}, \quad x = 0, 1, \dots, \quad (71)$$

- 1 The above modified q -Bessel distribution has Stieltjes-Wigert asymptotic behaviour for $n \rightarrow \infty$ as in (46) with

$$\mu_q = \frac{c}{1-q} h_n^{-1} \quad \text{and} \quad \sigma_q^2 = \frac{c^2}{q(1-q)} h_n^{-2} - \frac{c}{1-q} h_n^{-1},$$

where $0 < c < 1$, $h_n > 0$, $n = 0, 1, 2, \dots$ with $h_n = o(1)$.

- 2 If $q = q(n)$ with $q(n) \rightarrow 1$ as $n \rightarrow \infty$ and $q(n)^n = o(1)$ then, for $n \rightarrow \infty$, the modified q -Bessel distribution is approximated by a deformed standardized Gauss as in (57).
- 3 If $q = q(n)$ with $q(n) = 1 - \frac{\beta}{n}$, $0 < \beta \leq 1$ then, for $n \rightarrow \infty$, the modified q -Bessel distribution is approximated by a deformed standardized Gauss as in (68).

- **A Generalized q -Negative Binomial Distribution:** The q -CCVII for $b = q^n$, $n = 1, 2, \dots$ and $\eta = q$ becomes a generalized q -negative binomial distribution with p.f.

$$p(x) = P(X = x) = \frac{\binom{n+x-1}{x}_q q^{\binom{x}{2}} q^{-nx+x}}{\frac{(-q; q)_x}{(-q; q)_\infty}}, \quad x = 0, 1, \dots \quad (72)$$

- 1 The above q -negative binomial distribution has the Stieltjes-Wigert asymptotic behaviour for $n \rightarrow \infty$ as in (46) with

$$\mu_q = q^{-n+1} \frac{1 - q^n}{1 - q} \quad \text{and} \quad \sigma_q^2 = q^{-2n+1} \frac{1 - q^n}{q(1 - q)} + q^{-n+1} \frac{1 - q^n}{1 - q}.$$

- 2 If $q = q(n)$ with $q(n) \rightarrow 1$ as $n \rightarrow \infty$ and $q(n)^n = o(1)$ then, for $n \rightarrow \infty$, the q -negative binomial distribution is approximated by a deformed standardized Gauss as in (57).
- 3 If $q = q(n)$ with $q(n) = 1 - \frac{\beta}{n}$, $0 < \beta \leq 1$ then, for $n \rightarrow \infty$, the q -negative binomial distribution is approximated by a deformed standardized Gauss as in (68).

- **The Generalized Over / Underdispersed (O/U) Distribution:** The q -CCVII for $b = q^n$ and $\eta = \lambda q^n$ becomes a generalized O/U distribution with p.f.

$$p(x) = P(X = x) = \frac{(-\lambda q^n; q)_\infty}{(-\lambda; q)_\infty} \frac{(q^n; q)_x q^{\binom{x}{2}} \lambda^x}{(-\lambda q; q)_x (q; q)_x}, \quad x = 0, 1, \dots \quad (73)$$

(see Kemp(2005)).

- 1 The generalized O/U distribution for $\lambda = \lambda_n \rightarrow \infty$, has the Stieltjes-Wigert asymptotic behaviour for $n \rightarrow \infty$ as in (46) with $\mu_q = \lambda_n/(1-q)$ and $\sigma_q^2 = \frac{\lambda_n^2}{q(1-q)} + \frac{\lambda_n}{1-q}$.
- 2 If $q = q(n)$ with $q(n) \rightarrow 1$ as $n \rightarrow \infty$ and $q(n)^n = o(1)$ then, for $n \rightarrow \infty$, the generalized O/U distribution is approximated by a deformed standardized Gauss as in (57).
- 3 If $q = q(n)$ with $q(n) = 1 - \frac{\beta}{n}$, $0 < \beta \leq 1$ then, for $n \rightarrow \infty$, the generalized O/U distribution is approximated by a deformed standardized Gauss as in (68).

The discrete approximation of the q -CCV-III distribution for $n \rightarrow \infty$, is given by

$$p_{qCCVIII}(x) \cong (c; q)_{\infty} \frac{q^{2\binom{x}{2}} c^x}{(c; q)_x (q; q)_x}, \quad x = 0, 1, \dots \quad (74)$$

Berg and Valent (1994), have proved that for $q < a < 1/q$, the above discrete probability measure (74) has a continuous analogue counterpart family of absolutely continuous probability measures on $(0, \infty)$ defined by

$$v^{SC}(dx) = \frac{p}{\pi} \left\{ \left(\frac{a}{a-1} \frac{(x/a; q)_{\infty}^2}{(q/a; q)_{\infty}} \right)^2 + p^2 \left(\frac{(x; q)_{\infty}^2}{(q; q)_{\infty} (qa; q)_{\infty}} \right)^2 \right\}^{-1} dx, \quad (75)$$

where the parameter $p > 0$ is given by

$$p = \gamma / (t^2 + \gamma^2)$$

with

$$\gamma^2 = -t(1/\psi(a) + t),$$

where t belongs to the interval with endpoints 0 and $-1/\psi(a)$ and is given by

$$\psi(a) = (q; q)_{\infty} \sum_{j=0}^{\infty} \frac{q^j}{(a - q^j)(q; q)_j}$$

with $\psi(q^+) = \infty$.

Section 3: Concluding Remarks

- $0 < q < 1$ $q = q(n) \rightarrow 1, n \rightarrow \infty, q(n)^n = \Omega(1)$ or $q(n)^n = o(1)$



- Study of continuous limiting behaviour of confluent q -Chu-Vandermonde III distribution
- Study of continuous limiting behaviour of the members of families of non terminating and terminating q -Gauss series distributions

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Thank You!